

RIGID SCHUBERT VARIETIES IN COMPACT HERMITIAN SYMMETRIC SPACES

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ABSTRACT. Given a singular Schubert variety X_w in a compact Hermitian symmetric space X it is a longstanding question to determine when X_w is homologous to a smooth variety Y . We identify those Schubert varieties for which there exist first-order obstructions to the existence of Y . This extends (independent) work of M. Walters, R. Bryant and J. Hong.

Key tools include: (i) a new characterization of Schubert varieties that generalizes the well known description of the smooth Schubert varieties by connected sub-diagrams of a Dynkin diagram; and (ii) an algebraic Laplacian (à la Kostant), which is used to analyze the Lie algebra cohomology group associated to the problem.

1. INTRODUCTION

1.1. Motivation. Let X be an irreducible compact Hermitian symmetric space. The integral homology $H_*(X)$ is generated by the classes $[X_w]$ of the Schubert varieties $X_w \subset X$ [15, (6.4.5)]. (The Schubert varieties are indexed by elements w of the Hasse diagram associated to X ; see Section 2.3.) The majority of the Schubert varieties are singular. The following is a special case of a question posed by Borel and Haefliger in [2]: does there exist a smooth complex variety $Y \subset X$ that is homologous to X_w ; that is, $[Y] = [X_w]$?

Consider the case that $X = \text{Gr}(m, n)$ is the Grassmannian of m -planes in \mathbb{C}^n . The following examples are discussed in [3]. Given $k \leq n - m$, fix a subspace $W = \mathbb{C}^{n+1-m-k} \subset \mathbb{C}^n$. The set $\sigma(W) = \{E \in \text{Gr}(m, n) \mid E \cap W \neq 0\}$ is a codimension k Schubert variety. When $k = 1$, $\sigma(W)$ is singular, but homologous to a smooth variety. When $k = 2$, Hartshorne, Rees and Thomas prove that homology class of $\sigma(W)$ cannot be represented by any integral linear combination of smooth, oriented submanifolds in $\text{Gr}(3, 6)$ of (real) codimension four [8]. On the other hand, while $\sigma(W) \subset \text{Gr}(2, 5)$ is not homologous to a smooth variety, its homology class can be expressed as the difference of the homology classes of two smooth subvarieties.

Throughout, all (sub)manifolds are assumed to be connected, all (sub)varieties are assumed to be irreducible, and CHSS will denote an irreducible compact Hermitian symmetric space.

1.2. History. Given a complex variety $Y \subset X$, let $[Y] \in H_{2k}(X)$ denote its homology class, $\dim_{\mathbb{C}} Y = k$. The varieties Y satisfying $[Y] = r[X_w]$ for a positive integer r are characterized

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by the Schur differential system $\mathcal{R}_w \subset \text{Gr}(|w|, TX)$, with $|w| = \dim_{\mathbb{C}} X_w$; see Section 7.1. The Schubert variety X_w is *Schur rigid* if, for every integral variety Y of \mathcal{R}_w , there exists $g \in G$ such that $Y = g \cdot X_w$; otherwise we say X_w is *Schur flexible*. When X_w is Schur rigid the only complex subvarieties of X with homology class $r[X_w]$ are the G -translates $g \cdot X_w$. Therefore

if the Schubert variety X_w is singular and Schur rigid, then there exist no smooth varieties that are homologous to X_w .

The Schur system was first studied independently by Bryant [3] and M. Walters [17]. Walters studied the Schur rigidity of codimension two Schubert varieties in $\text{Gr}(2, n)$, and smooth Schubert varieties in $\text{Gr}(m, n)$. Bryant obtained similar results in addition to studying some of the singular Schubert varieties in $\text{Gr}(m, n)$, the smooth Schubert varieties in the Lagrangian grassmannians, and the maximal linear subspaces (which are smooth Schubert varieties) in the classical CHSS.

In the case that X_w is smooth, the results of Bryant and Walters were generalized to arbitrary CHSS and given a uniform proof by J. Hong.

Theorem (Hong [10]). *Let X be an irreducible compact Hermitian symmetric space in its minimal homogeneous embedding, excluding the quadrics of odd dimension. Let $X_w \subset X$ be a smooth Schubert variety, excluding the non-maximal linear subspaces and $\mathbb{P}^1 \subset C_n/P_n$. Then X_w is Schur rigid.*[†]

It is straight-forward to see that the cases excluded from Hong's theorem are not Schur rigid: when $H_{2|w|}(X)$ is generated by a single Schubert variety X_w , every $|w|$ -dimensional subvariety $Y \subset X$ must satisfy $[Y] = r[X_w]$ for some $r \in \mathbb{Z}_{>0}$. Thus, every $|w|$ -dimensional Y is an integral variety of \mathcal{R}_w . This is the case when X is a projective space or an odd dimensional quadric hypersurface. For similar reasons, we can rule out the cases where X_w is a non-maximal linear space or $X_w = \mathbb{P}^1 \subset C_n/P_n$.

Making use of [10] and the foliation structure of Schubert varieties, Hong also proved that a large class of the singular Schubert varieties in the Grassmannian are Schur rigid [9].

There are variants of this ‘smoothing problem.’ For example, I. Choe and Hong have studied the related notion of Schubert rigidity of linear subspaces of arbitrary homogeneous varieties with second betti number equal to one [6]. In another direction, S. Kleiman considered the problem of deforming a cycle Z in an arbitrary projective variety by rational equivalence into the difference of two effective cycles $Z_1 - Z_2$ whose prime components are smooth. In [12] Kleiman and J. Landolfi specialized the problem to case the projective variety is a Grassmannian.

1.3. Strategy. The problem is approached as follows. The bundle \mathcal{R}_w contains the sub-bundle \mathcal{B}_w of $|w|$ -planes tangent to a smooth point of $g \cdot X_w$, for some $g \in G$. When the only integral varieties of \mathcal{B}_w are $g \cdot X_w$, we say X_w is *Schubert rigid*; otherwise X_w is *Schubert flexible*. Hong proved that X_w is Schur rigid if and only if X_w is Schubert rigid and $\mathcal{B}_w = \mathcal{R}_w$ [10]. Because the Schubert system \mathcal{B}_w is more amenable to analysis than the Schur system \mathcal{R}_w , the general approach to this problem is to first show that X_w is Schubert rigid, and then prove that $\mathcal{B}_w = \mathcal{R}_w$.

[†]The maximal linear space $\mathbb{P}^1 \subset C_n/P_n$ was accidentally omitted in Hong's theorem.

The key observation in Hong’s proof for the smooth Schubert varieties is that the (partial) vanishing of a certain Lie algebra cohomology group implies the Schubert rigidity of X_w . When the Schubert variety is smooth, the cohomology group satisfies the hypotheses of Kostant’s famous theorem [14, Theorem 5.14], and it is straightforward to determine when the vanishing holds. Hong then directly computes $\mathcal{B}_w = \mathcal{R}_w$, establishing Schur rigidity.

The key difficulty in extending Hong’s approach to the singular Schubert varieties is the absence of a Kostant-type theorem for the associated Lie algebra cohomology. (We do not see a natural extension of Hong’s method in the case that $X = \text{Gr}(m, n)$ to arbitrary CHSS.)

1.4. Contents. The main result of this paper is Theorem 8.1 which identifies the Schubert varieties for which there exist first-order obstructions to Schur flexibility. The theorem recovers (and extends) the results of Walters, Bryant and Hong to the general case. The varieties are described in terms of a new characterization of the X_w by a nonnegative integer $\mathbf{a}(w)$ and a marking $J(w)$ of the Dynkin diagram of G (Proposition 3.9 and Corollary 3.17). This description is the *sine qua non* of our analysis of the Schubert system \mathcal{B}_w and the equality $\mathcal{B}_w = \mathcal{R}_w$. It generalizes the well-known characterization of the smooth Schubert varieties by connected sub-diagrams of the Dynkin diagram of G – the smooth X_w correspond to $\mathbf{a}(w) = 0$ (Proposition 3.19).

Sections 2 and 3 present the necessary definitions and background on homogeneous varieties, their Schubert subvarieties and compact Hermitian symmetric spaces. In Section 4 the Schubert system is lifted to a frame bundle where the analysis is performed. To each X_w there is associated a Lie algebra cohomology group $H^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp)$ whose vanishing (in positive degree) ensures Schubert rigidity. Following the constructions of Kostant in [14] we define a Laplacian \square in Section 5 and show that there is a natural bijection between $H^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp)$ and $\ker \square =: \mathcal{H}^1$ (Proposition 5.10).

There is a reductive Lie algebra $\mathfrak{g}_{0,0} \subset \mathfrak{g}$ with respect to which the Laplacian acts as a $\mathfrak{g}_{0,0}$ -module morphism $\square : \mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^* \rightarrow \mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^*$. Schur’s Lemma implies that $\mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^*$ admits a $\mathfrak{g}_{0,0}$ -module decomposition into \square -eigenspaces. (The eigenvalues are non-negative.) Through analysis of the spectrum of \square we see that the desired vanishing occurs if and only if two representation theoretic conditions hold. In particular, X_w is Schubert rigid when the two conditions are satisfied (Theorem 5.38).

Theorem 6.1 lists the corresponding Schubert rigid X_w . This completes the first step of the approach to the Schur rigidity problem. It remains to determine when $\mathcal{B}_w = \mathcal{R}_w$. In Section 7 we develop a test (7.10) to determine when $\mathcal{B}_w = \mathcal{R}_w$ holds. In Section 8 we apply the test to the varieties of Theorem 6.1; we find that they are all Schur rigid (Theorem 8.1).

The present paper leaves a question to address. The Schubert varieties listed in Theorem 6.1 are those for which there exist first-order obstructions to the existence of nontrivial integral varieties of the Schur system. (An integral variety is *trivial* if it is of the form $g \cdot X_w$ for some $g \in G$.) In particular, if X_w is not listed in Theorem 6.1, it does not immediately follow that X_w is Schur flexible: there may exist higher-order obstructions to the existence of nontrivial integral varieties. It remains to determine which of these X_w are flexible. This problem will be addressed in a sequel.

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2. HOMOGENEOUS VARIETIES

2.1. Notation. We employ the root and weight conventions of [13].

Let \mathfrak{g} be a complex semi-simple Lie algebra. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, and let \mathfrak{h} be the associated Cartan subalgebra. Let Δ denote the roots of \mathfrak{g} , and Δ^{\pm} the positive and negative roots. Given $\beta \in \Delta$, let $\mathfrak{g}_{\beta} \subset \mathfrak{g}$ denote the corresponding root space. Given a direct sum $\mathfrak{s} \subset \mathfrak{g}$ of root spaces, let $\Delta(\mathfrak{s}) \subset \Delta$ denote the corresponding roots. That is, $\Delta(\mathfrak{s})$ is defined by

$$\mathfrak{s} = \bigoplus_{\beta \in \Delta(\mathfrak{s})} \mathfrak{g}_{\beta}.$$

Fix simple roots $\{\alpha_1, \dots, \alpha_n\} = \Sigma \subset \Delta^+$. Let $\mathfrak{p} \supset \mathfrak{b}$ denote the parabolic subalgebra generated by a subset $\Sigma_{\mathfrak{p}} \subset \Sigma$. For example, $\Sigma_{\mathfrak{b}} = \emptyset$, and $\Sigma_{\mathfrak{p}} = \{\alpha_1, \dots, \hat{\alpha}_{\mathbf{i}}, \dots, \alpha_n\}$ generates a maximal parabolic. Let $I_{\mathfrak{p}} = \{i \mid \alpha_i \in \Sigma_{\mathfrak{p}}\}$ denote the corresponding index set.

Given a dominant integral weight ν of \mathfrak{g} , let V_{ν} denote the unique irreducible \mathfrak{g} -representation of highest weight ν . Let $\{\omega_1, \dots, \omega_n\}$ denote the fundamental weights of \mathfrak{g} and

$$\rho := \sum_{i=1}^n \omega_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \quad \text{and} \quad \rho_{\mathfrak{p}} := \sum_{i \notin I_{\mathfrak{p}}} \omega_i.$$

For example, if \mathfrak{p} is maximal, $\rho_{\mathfrak{p}} = \omega_{\mathbf{i}}$. For the Borel subalgebra, $\rho_{\mathfrak{b}} = \rho$.

Let $P \subset G$ be connected, complex semi-simple Lie groups with Lie algebras $\mathfrak{p} \subset \mathfrak{g}$. The G -orbit $X \subset \mathbb{P}V_{\rho_{\mathfrak{p}}}$ of the highest weight line in $V_{\rho_{\mathfrak{p}}}$ is the *minimal homogeneous embedding* of G/P . Let $o \in X$ denote the highest weight line. As a \mathfrak{p} -module,

$$(2.1) \quad T_o X = \mathfrak{g}/\mathfrak{p}.$$

Throughout, $g \cdot$ will denote the action of $g \in G$, and $\xi \cdot$ the action of $\xi \in \mathfrak{g}$.

2.2. Grading elements. The parabolic \mathfrak{p} determines a *grading element* $Z = Z_{\mathfrak{p}} \in \mathfrak{h}$ by

$$\alpha_i(Z) = \begin{cases} 0, & \alpha_i \in \Sigma_{\mathfrak{p}}, \\ 1, & \alpha_i \notin \Sigma_{\mathfrak{p}}. \end{cases}$$

Let $\mathfrak{g}_j = \{u \in \mathfrak{g} \mid [Z, u] = j u\}$ be the Z -eigenspace with eigenvalue j . Then

$$\mathfrak{g} = \underbrace{\mathfrak{g}_q \oplus \cdots \oplus \mathfrak{g}_1}_{\mathfrak{g}_+} \oplus \mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-q}}_{\mathfrak{g}_-}$$

is a graded decomposition. That is, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In particular, each \mathfrak{g}_j is a \mathfrak{g}_0 -module. Note that $\mathfrak{h} \subset \mathfrak{g}_0$ and $\mathfrak{p} = \mathfrak{g}_{\geq 0}$. So (2.1) yields a natural identification

$$(2.2) \quad T_o X = \mathfrak{g}_-$$

as \mathfrak{g}_0 -modules. Additionally, \mathfrak{g}_0 is a reductive Lie subalgebra of \mathfrak{g} . Indeed

$$\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{f},$$

where $\mathfrak{z} \subset \mathfrak{h}$ is the center of \mathfrak{g}_0 and \mathfrak{f} is the semi-simple subalgebra of \mathfrak{g} with simple roots $\Sigma_{\mathfrak{p}}$. Let $\{Z_i\}$ be the basis of \mathfrak{h} dual to the root basis $\{\alpha_i\}$ of \mathfrak{h}^* . Then $\mathfrak{z} = \text{span}\{Z_i \mid i \notin I_{\mathfrak{p}}\}$, and the grading element $Z = Z_{\mathfrak{p}}$ is the sum $\sum_{i \notin I_{\mathfrak{p}}} Z_i$.

2.3. The Hasse diagram of \mathfrak{p} . Let W denote the Weyl group of G and let $|w|$ denote the length of $w \in W$. Define

$$(2.3) \quad \Delta(w) := w\Delta^- \cap \Delta^+.$$

Definition 2.4. The *Hasse diagram* $[1, 5]$ of P is

$$\begin{aligned} W^{\mathfrak{p}} &:= \{w \in W \mid \Delta^+(\mathfrak{g}_0) \subset w(\Delta^+)\} = \{w \in W \mid \Delta(w) \subset \Delta(\mathfrak{g}_+)\} \\ &= \{w \in W \mid w(\lambda) \text{ is } \mathfrak{g}_0\text{-dominant } \forall \mathfrak{g}\text{-dominant weights } \lambda\}. \end{aligned}$$

Definition. A set $\Phi \subset \Delta$ is *closed* or *saturated* if given any two $\beta, \gamma \in \Phi$ such that $\beta + \gamma \in \Delta$, it is the case that $\beta + \gamma \in \Phi$.

Proposition 2.5. *The mapping $w \mapsto \Delta(w)$ is a bijection of $W^{\mathfrak{p}}$ onto the family of all subsets $\Phi \subset \Delta(\mathfrak{g}_+)$ with the property that both Φ and $\Delta^+ \setminus \Phi$ are closed.*

For the proof, see [14, Proposition 5.10] or [5, Proposition 3.2.12].

Remark. We cite [14, 15, 5] often in this paper. We note that our $\Delta(w)$ is their Φ_w .

2.4. Schubert varieties. Let $B = \exp(\mathfrak{b}) \subset G$ denote the Borel subgroup with Lie algebra \mathfrak{b} . The Schubert varieties X_w of X are indexed by $w \in W^{\mathfrak{p}}$. Their homology classes $[X_w]$ form a basis for the integral homology $H_*(X)$. Let $C_w := Bw^{-1} \cdot o \subset X$ denote the *Schubert cell*. Then $X = \bigcup_{w \in W^{\mathfrak{p}}} C_w$ is a disjoint union. Let $X_w := \overline{C_w} \subset X$ denote the *Schubert variety*. More generally, any $g \cdot X_w$, where $g \in G$, is a Schubert variety in X . We will abuse notation by referring to any of these varieties as X_w .

The cell $w \cdot C_w$ is the orbit $N_w \cdot o$ of a unipotent subgroup $N_w \subset G$. This is seen as follows. Let $N \subset B$ be the maximal unipotent subgroup of the Borel. Let $w_0 \in W$ be the longest element and $N^- := w_0 N w_0^{-1} \subset G$ the unipotent subgroup ‘opposite’ to N . Then $N_w := w N w^{-1} \cap N^-$. The Lie algebra of N_w is

$$(2.6) \quad \mathfrak{n}_w := \bigoplus_{\alpha \in \Delta(w)} \mathfrak{g}_{-\alpha}.$$

It is well-known that $|\Delta(w)| = |w|$ ([5, Proposition 3.2.14 (3)]). Thus the Schubert variety X_w is of dimension $|w|$.

2.5. Conjugation and duality. Recall that any automorphism $\varphi : \delta_{\mathfrak{g}} \rightarrow \delta_{\mathfrak{g}}$ of the Dynkin diagram induces automorphisms $\varphi : \Delta \rightarrow \Delta$ and $\varphi : W \rightarrow W$ of the root system and Weyl group. The latter is given as follows: if σ_j denotes the reflection corresponding to the simple root α_j , and $w = \sigma_{i_1} \cdots \sigma_{i_r}$, then $\varphi(w)$ is $\sigma_{\varphi(i_1)} \cdots \sigma_{\varphi(i_r)}$.

Definition. Given $w \in W$ and an automorphism $\varphi : \delta_{\mathfrak{g}} \rightarrow \delta_{\mathfrak{g}}$, let $w' = \varphi(w)$ denote the φ -conjugate.

Remarks. It is clear from Definition 2.4 that $w \in W^{\mathfrak{p}_1}$ if and only if $w' \in W^{\mathfrak{p}_{\varphi(1)}}$. The group of Dynkin diagram automorphisms is \mathfrak{S}_3 for D_4 ; \mathbb{Z}_2 for A_n , D_n ($n > 4$) and E_6 ; trivial for the remaining complex simple Lie groups.

The Weyl group of the semi-simple part \mathfrak{f} of \mathfrak{g}_0 may be identified with the subgroup $W_{\mathfrak{p}} \subset W$ generated by $\{\sigma_j\}_{j \in I_{\mathfrak{p}}}$. Let $w_{\mathfrak{p}}^0$ be the longest word in $W_{\mathfrak{p}}$. Then

$$(2.7) \quad w_{\mathfrak{p}}^0(\Delta^+(\mathfrak{g}_0)) = \Delta^-(\mathfrak{g}_0) \quad \text{and} \quad w_{\mathfrak{p}}^0(\Delta(\mathfrak{g}_+)) = \Delta(\mathfrak{g}_+).$$

Definition. The dual of $w \in W$ is $w^* = w_{\mathfrak{p}}^0 w w_0$.

Note that $\varphi(w_{\mathfrak{p}}^0 w w_0) = \varphi(w_{\mathfrak{p}}^0) \varphi(w) \varphi(w_0) = \omega_{\varphi(\mathfrak{p})}^0 \varphi(w) w_0$ implies

$$(w^*)' = (w')^*.$$

Above, $\varphi(\mathfrak{p})$ is the parabolic subalgebra with $I_{\varphi(\mathfrak{p})} = \varphi(I_{\mathfrak{p}})$.

Lemma 2.8.

- (a) *Duality is an involution* $(w^*)^* = w$;
- (b) $w \in W^{\mathfrak{p}}$ if and only if $w^* \in W^{\mathfrak{p}}$;
- (c) $\Delta(w^*) = w_{\mathfrak{p}}^0(\Delta(\mathfrak{g}_+) \setminus \Delta(w)) = \Delta(\mathfrak{g}_+) \setminus w_{\mathfrak{p}}^0 \Delta(w)$ for any $w \in W^{\mathfrak{p}}$.

Proof. (a) is a consequence of $(w_0)^{-1} = w_0$ and $(w_{\mathfrak{p}}^0)^{-1} = w_{\mathfrak{p}}^0$.

To prove (b), let (\cdot, \cdot) denote the Killing form on \mathfrak{h}^* , λ a \mathfrak{g} -dominant weight and $\alpha \in \Delta^+(\mathfrak{g}_0)$. Then $(w^*(\lambda), \alpha) = (\lambda, (w^*)^{-1}(\alpha)) = (\lambda, w_0 w^{-1} w_{\mathfrak{p}}^0(\alpha))$. Since $w \in W^{\mathfrak{p}}$, we have $w^{-1}(\Delta^+(\mathfrak{g}_0)) \subset \Delta^+$. Thus, $w^{-1} w_{\mathfrak{p}}^0(\alpha) \in \Delta^-$ and hence $w_0 w^{-1} w_{\mathfrak{p}}^0(\alpha) \in \Delta^+$. From the

\mathfrak{g} -dominance of λ we conclude $(w^*(\lambda), \alpha) \geq 0$. Thus, $w \in W^{\mathfrak{p}}$. Conversely, if $w^* \in W^{\mathfrak{p}}$, then $w = (w^*)^* \in W^{\mathfrak{p}}$.

The proof of (c) requires the identities

- (i) $\Delta(w w_0) = \Delta^+ \setminus \Delta(w)$ for any $w \in W$;
- (ii) $w_{\mathfrak{p}}^0 \Delta(w) = \Delta(w_{\mathfrak{p}}^0 w) \setminus \Delta^+(\mathfrak{g}_0)$ for any $w \in W^{\mathfrak{p}}$.

The first identity is proved in [5, p. 324]. To prove the second identity, note that (2.3), Definition 2.4 and (2.7) imply $w_{\mathfrak{p}}^0 \Delta(w) \subset \Delta(\mathfrak{g}_+)$ and $\Delta(w) = w \Delta^- \cap \Delta(\mathfrak{g}_+)$. Therefore, $w_{\mathfrak{p}}^0 \Delta(w) = w_{\mathfrak{p}}^0 (w \Delta^- \cap \Delta(\mathfrak{g}_+)) = w_{\mathfrak{p}}^0 w \Delta^- \cap \Delta(\mathfrak{g}_+) = \Delta(w_{\mathfrak{p}}^0 w) \setminus \Delta^+(\mathfrak{g}_0)$. Hence,

$$\begin{aligned} \Delta(\mathfrak{g}_+) \setminus w_{\mathfrak{p}}^0 \Delta(w) &\stackrel{(ii)}{=} \Delta(\mathfrak{g}_+) \setminus (\Delta(w_{\mathfrak{p}}^0 w) \setminus \Delta^+(\mathfrak{g}_0)) = (\Delta^+ \setminus \Delta(w_{\mathfrak{p}}^0 w)) \setminus \Delta^+(\mathfrak{g}_0) \\ &\stackrel{(i)}{=} \Delta(w_{\mathfrak{p}}^0 w w_0) \setminus \Delta^+(\mathfrak{g}_0) = \Delta(w^*) \setminus \Delta^+(\mathfrak{g}_0) = \Delta(w^*); \end{aligned}$$

the final equality is a consequence of part (b) of the lemma, and Definition 2.4. \square

Given $\alpha \in \Delta$, define $\alpha^* := -w_{\mathfrak{p}}^0(\alpha)$. It follows from (2.7) that $\Delta(\mathfrak{g}_0)^* = \Delta(\mathfrak{g}_0)$. Recall that $w_{\mathfrak{p}}^0(\Sigma_{\mathfrak{p}}) = -\Sigma_{\mathfrak{p}}$, cf. [5, p. 324]. This induces $*$: $I_{\mathfrak{p}} \rightarrow I_{\mathfrak{p}}$ mapping $j \mapsto j^*$ by

$$(2.9) \quad (\alpha_j)^* = -w_{\mathfrak{p}}^0(\alpha_j) = \alpha_{j^*}.$$

Remark 2.10. Since W preserves the Killing form (\cdot, \cdot) , we have $(\alpha^*, \beta^*) = (\alpha, \beta)$ for any $\alpha, \beta \in \Delta$. This implies that the map $*$: $I_{\mathfrak{p}} \rightarrow I_{\mathfrak{p}}$ corresponds to a Dynkin diagram automorphism of the subgraph $\delta_{\mathfrak{p}} \subset \delta_{\mathfrak{g}}$ generated by $I_{\mathfrak{p}}$.

3. COMPACT HERMITIAN SYMMETRIC SPACES

3.1. Definition. The irreducible compact Hermitian symmetric spaces (CHSS, Table 1) are those G/P with G simple and graded decomposition (cf. §2.2)

$$(3.1) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

In particular, the nilpotent subalgebras $\mathfrak{g}_+ = \mathfrak{g}_1$ and $\mathfrak{g}_- = \mathfrak{g}_{-1}$ are abelian

$$(3.2) \quad [\mathfrak{g}_1, \mathfrak{g}_1] = \{0\} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}].$$

The parabolic P is always maximal. So $\Sigma \setminus \Sigma_{\mathfrak{p}}$ consists of a single simple root α_1 and $V_{\rho_{\mathfrak{p}}} = V_{\omega_1}$.

TABLE 1. The CHSS.

Classical CHSS	Grassmannians $\text{Gr}(\mathfrak{i}, n+1) = A_n/P_1$.
	Quadric hypersurfaces B_n/P_1 with $n \geq 2$, and D_n/P_1 with $n \geq 4$.
	Lagrangian grassmannians C_n/P_n with $n \geq 3$.
	Spinor varieties $D_n/P_{n-1} \simeq D_n/P_n$ with $n \geq 4$.
Exceptional CHSS	The Cayley plane $E_6/P_1 \simeq E_6/P_6$.
	The Freudenthal variety E_7/P_7 .

3.2. Characterization of \mathfrak{n}_w for CHSS. Let $Z_{\mathbf{i}}$ be the grading element associated to the CHSS $G/P_{\mathbf{i}}$ (§2.2), and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the $Z_{\mathbf{i}}$ -graded decomposition of \mathfrak{g} (§3.1). Let $\tilde{\alpha}$ denote the highest root of $\Delta(\mathfrak{g}_1)$.

Given $J \subset I_{\mathbf{p}} = \{1, \dots, n\} \setminus \{\mathbf{i}\}$, define $Z_J = \sum_{s \in J} Z_s$. Let $\mathfrak{g} = \oplus \mathfrak{g}_{j,k}$ be the $(Z_{\mathbf{i}}, Z_J)$ -bigraded decomposition of \mathfrak{g} : that is,

$$(3.3) \quad \mathfrak{g}_{j,k} := \{u \in \mathfrak{g} \mid [Z_{\mathbf{i}}, u] = j u, [Z_J, u] = k u\}.$$

Lemma 3.4. *Given an integer $0 \leq a \leq \tilde{\alpha}(Z_J)$, there exists a unique $w = w(J, a) \in W^{\mathbf{p}}$ such that $\mathfrak{g}_{-1,0} \oplus \dots \oplus \mathfrak{g}_{-1,-a} = \mathfrak{n}_w$.*

Proof. Define $\Phi = \Delta(\mathfrak{g}_{1,0} \oplus \dots \oplus \mathfrak{g}_{1,a}) = \{\alpha \in \Delta \mid \alpha(Z_{\mathbf{i}}) = 1, \alpha(Z_J) \leq a\} \subset \Delta(\mathfrak{g}_1)$. We will show that $\Phi = \Delta(w)$ for some $w \in W^{\mathbf{p}}$. By Proposition 2.5 it suffices to show that Φ and $\Delta^+ \setminus \Phi$ are closed. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = \{0\}$, every subset of $\Delta(\mathfrak{g}_1)$ is closed. Hence $\Phi = \Delta(w)$ for some $w \in W^{\mathbf{p}}$ if and only if $\Delta^+ \setminus \Phi$ is closed. This follows immediately from $\Delta^+ \setminus \Phi = \Delta(\mathfrak{g}_{1,>a}) \sqcup \Delta^+(\mathfrak{g}_{0,\geq 0})$ and $[\mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,>a}, \mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,>a}] \subset \mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,>a}$. Thus, $w = w(J, a)$ exists and moreover it is unique by Proposition 2.5. \square

We will see below that Lemma 3.4 has a converse. We identify $\Sigma_{\mathbf{p}} = \{\alpha_j \mid j \neq \mathbf{i}\}$ with the simple roots of \mathfrak{f} . The stabilizer $\mathfrak{q}^w \subset \mathfrak{f}$ of $\mathfrak{n}_w \in \text{Gr}(|w|, \mathfrak{g}_{-1})$ is a parabolic subalgebra. (See Section 4.1.) Let $I_{\mathfrak{q}^w} = I_w \subset I_{\mathbf{p}}$ be the corresponding index set (§2.1), and set

$$J = J(w) = I_{\mathbf{p}} \setminus I_w.$$

Note that

$$(3.5) \quad j \in J \iff \mathfrak{g}_{-\alpha_j} \text{ does not stabilize } \mathfrak{n}_w.$$

Remark 3.6. Note that $J = \emptyset$ if and only if $X_w = o$ or $X_w = X$. We will assume that X_w is a proper subvariety of X .

Define

$$(3.7) \quad Z_w := Z_J = \sum_{j \in J} Z_j \quad \text{and} \quad \mathbf{a} = \mathbf{a}(w) := \max\{\alpha(Z_w) \mid \alpha \in \Delta(w)\} \in \mathbb{Z}_{\geq 0}.$$

The $(Z_{\mathbf{i}}, Z_w)$ -bigraded decomposition $\mathfrak{g} = \oplus \mathfrak{g}_{j,k}$ is given by (3.3) and

$$(3.8) \quad \mathfrak{g}_{0,\geq 0} = \mathfrak{z} \oplus \mathfrak{q}^w \text{ is the stabilizer in } \mathfrak{g}_0 \text{ of } \mathfrak{n}_w \subset \mathfrak{g}_{-1}.$$

We may now state the converse to Lemma 3.4.

Proposition 3.9. *Let $G/P_{\mathbf{i}}$ be an irreducible compact Hermitian symmetric space. Given $w \in W^{\mathbf{p}}$, let $\mathfrak{g} = \oplus \mathfrak{g}_{j,k}$ be the $(Z_{\mathbf{i}}, Z_w)$ -bigraded decomposition (3.3) of \mathfrak{g} . Then*

$$(3.10) \quad \mathfrak{n}_w = \mathfrak{g}_{-1,0} \oplus \dots \oplus \mathfrak{g}_{-1,-\mathbf{a}(w)}.$$

See Corollary 3.17 for a description of the (\mathbf{a}, J) pairs that occur. The smooth X_w are characterized by $\mathbf{a}(w) = 0$ (Proposition 3.19).

The proof of the proposition is given in five lemmas. The final lemma is proved in a case-by-case argument for each of the classical CHSS (Table 1). We used the representation theory software LiE [16] to confirm Proposition 3.9 for the two exceptional cases. The first lemma is due to Kostant. See [18, Theorem 8.13.3] for a more general statement and proof.

Lemma 3.11 (Kostant). *Let \mathfrak{a} be a complex semi-simple Lie algebra with a choice of Cartan subalgebra \mathfrak{t} and simple roots $\{\alpha_1, \dots, \alpha_n\}$. Define $Z_i \in \mathfrak{t}$ by $\alpha_i(Z_j) = \delta_{ij}$. Any $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ defines a multi-graded decomposition $\mathfrak{a} = \bigoplus_{A \in \mathbb{Z}^p} \mathfrak{a}_A$ by $\mathfrak{a}_A = \mathfrak{a}_{a_1, \dots, a_p} := \{u \in \mathfrak{a} \mid [Z_{i_m}, u] = a_m u, \forall 1 \leq m \leq p\}$. Each $\mathfrak{a}_A \neq \mathfrak{a}_{0, \dots, 0} =: \mathfrak{a}_0$ is an irreducible \mathfrak{a}_0 -module.*

Define

$$\{j_1, j_2, \dots, j_p\} := J \quad \text{with} \quad j_1 < j_2 < \dots < j_p.$$

Let $\mathfrak{g}_{-1} = \bigoplus_A \mathfrak{g}_{-1, -A}$ denote the multi-graded decomposition induced by $(Z_{j_1}, \dots, Z_{j_p})$. Since the roots $\Delta(\mathfrak{g}_{-1})$ are non-positive integer linear combinations of the simple roots, we have $A = (a_1, \dots, a_p) \geq (0, \dots, 0)$. Define

$$|A| := a_1 + \dots + a_p.$$

Note that $|A| = \alpha(Z_w)$ for all $\alpha \in \Delta(\mathfrak{g}_{1, A})$, so that $\mathfrak{g}_{1, A} \subset \mathfrak{g}_{1, |A|}$. Let

$$\mathfrak{n}_w^\perp := \bigoplus_{\alpha \in \Delta(\mathfrak{g}_1) \setminus \Delta(w)} \mathfrak{g}_{-\alpha}.$$

Definition 3.12. Define $B \leq A$ if there exists a sequence of $\mathfrak{g}_{0, -C_j} \subset \mathfrak{g}_{0, -1}$ such that $\mathfrak{g}_{1, B} = [\mathfrak{g}_{0, -C_1}, [\mathfrak{g}_{0, -C_2}, \dots [\mathfrak{g}_{0, -C_r}, \mathfrak{g}_{1, A}] \dots]]$. That is, $\mathfrak{g}_{1, B}$ may be obtained from $\mathfrak{g}_{1, A}$ by successive brackets with the irreducible $\mathfrak{g}_{0, 0}$ -submodules $\mathfrak{g}_{0, -C}$ of $\mathfrak{g}_{0, -1}$.

Lemma 3.13. *If $\mathfrak{g}_{-1, -A} \subset \mathfrak{n}_w$ and $B < A$, then $\mathfrak{g}_{-1, -B} \subset \mathfrak{n}_w$.*

Proof. It suffices to consider the case that $\mathfrak{g}_{1, B} = [\mathfrak{g}_{0, -C}, \mathfrak{g}_{1, A}]$; the general case will follow by induction. By Kostant's Lemma 3.11, $\mathfrak{g}_{-1, -B}$ is an irreducible $\mathfrak{g}_{0, 0}$ -module. So either $\mathfrak{g}_{-1, -B} \subset \mathfrak{n}_w$, or $\mathfrak{g}_{-1, -B} \subset \mathfrak{n}_w^\perp$. Observe that $[\mathfrak{g}_{0, C}, \mathfrak{g}_{-1, -A}] = \mathfrak{g}_{-1, -B}$ if and only if $[\mathfrak{g}_{0, -C}, \mathfrak{g}_{-1, -B}] = \mathfrak{g}_{-1, -A}$. It follows from Proposition 2.5 that $\mathfrak{g}_{-1, -B} \subset \mathfrak{n}_w$. \square

Lemma 3.14. *Define $\mathfrak{m} = \tilde{\alpha}(Z_w)$, where $\tilde{\alpha}$ is the highest root of $\Delta(\mathfrak{g}_1)$.*

- (a) *If $\beta \in \Delta(\mathfrak{g}_1)$ and $\beta(Z_w) < \mathfrak{m}$, then $[\mathfrak{g}_{0, 1}, \mathfrak{g}_\beta] \neq \{0\}$.*
- (b) *If $0 \leq b < \mathfrak{m}$, then $\mathfrak{g}_{1, b} = [\mathfrak{g}_{0, -1}, \mathfrak{g}_{1, b+1}]$.*

Proof. Argue by contradiction: suppose that $[\mathfrak{g}_{0, 1}, \mathfrak{g}_\beta] = \{0\}$. Equivalently, the intersection of $\Delta(\mathfrak{g}_{1, b+1})$ with $\Delta(\mathfrak{g}_{0, 1}) + \beta$ is empty. This implies $\mathfrak{g}_\beta \not\subset [\mathfrak{g}_{0, -1}, \mathfrak{g}_{1, b+1}]$, where $b = |B|$. This is a contradiction as the root space \mathfrak{g}_β is obtained from the highest root space $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{g}_{1, \mathfrak{m}}$ by successive brackets with the $\{\mathfrak{g}_{-\alpha_j} \mid \alpha_j \text{ a simple root}\}$. This establishes (a) and (b). \square

The following lemma will establish Proposition 3.9.

Lemma 3.15. *Assume G/P is one of the classical CHSS listed (cf. Table 1). If $|B| \leq \mathfrak{a}$, then $\mathfrak{g}_{-1, -B} \subset \mathfrak{n}_w$.*

Proof. From Lemma 3.14 we see that if $\mathfrak{g}_{1, B} \subset \mathfrak{g}_1$ and $|B| < \mathfrak{a}$, then there exists $\mathfrak{g}_{1, B'} \subset \mathfrak{g}_1$ with $B < B'$ and $|B'| = \mathfrak{a}$. So it follows from Lemma 3.13 that it suffices to show that $|B| = \mathfrak{a}$ implies $\mathfrak{g}_{-1, -B} \in \mathfrak{n}_w$.

We will argue by contradiction, showing that the existence of $\mathfrak{g}_{-1, -B} \not\subset \mathfrak{n}_w$ with $|B| = \mathfrak{a}$ implies that \mathfrak{n}_w is stabilized by a simple root space $\mathfrak{g}_{-\alpha_\ell} \subset \mathfrak{g}_{0, -}$ that is not contained in the stabilizer $\mathfrak{g}_{0, \geq 0}$ of \mathfrak{n}_w in \mathfrak{g}_0 . The reader should be aware that the argument implicitly makes (very) frequent use of Lemma 3.13, often without mention. The proof proceeds case-by-case

through the classical CHSS. The assertions below regarding the expressions for $\alpha \in \Delta(\mathfrak{g}_1)$ as integral linear combinations of the simple roots $\{\alpha_1, \dots, \alpha_n\}$ may be found in standard representation theory texts.

For notational convenience we will use superscripts to write multi-indices compactly. For example,

$$(0^k, 1^\ell, 2^m) := (\underbrace{0, \dots, 0}_{k \text{ terms}}, \underbrace{1, \dots, 1}_{\ell \text{ terms}}, \underbrace{2, \dots, 2}_{m \text{ terms}}).$$

(I) We begin with B_n/P_1 . The roots $\alpha \in \Delta(\mathfrak{g}_1)$ are of the form $\alpha = \alpha_1 + \dots + \alpha_j$ or $\alpha = \alpha_1 + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_n)$, with $1 < j \leq n$. Thus the $B = (b_1, \dots, b_p)$ with $|B| = \mathfrak{a}$ are of the form $(1^{\mathfrak{a}}, 0^{\mathfrak{p}-\mathfrak{a}})$ or $(1^{\mathfrak{a}-2m}, 2^m)$, with $2m \leq \mathfrak{a}$. Observe that $\mathfrak{p} \leq \mathfrak{a} + 1$, else Lemma 3.13 and some thought imply $\mathfrak{g}_{-\alpha_{j\mathfrak{p}}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w . Similarly, $\mathfrak{a} \leq \mathfrak{p}$, else $\mathfrak{g}_{-\alpha_{j\mathfrak{p}}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w .

If $\mathfrak{p} = \mathfrak{a}$, then $B = (1^{\mathfrak{a}})$ is uniquely determined and the lemma follows, in this case, from an application of Lemma 3.13. Moreover, if $j \in J - \{j_{\mathfrak{p}}\}$, then $\mathfrak{g}_{-\alpha_j} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Therefore $\mathfrak{p} = 1$, so that $J = \{j_1\}$, and $\mathfrak{a} = 1$.

If $\mathfrak{p} = \mathfrak{a} + 1$, then $B = (1^{\mathfrak{a}}, 0)$ is uniquely determined and the lemma again follows from Lemma 3.13. As in the previous paragraph, we also see that $J = \{j_1\}$ and $\mathfrak{a} = 0$.

(II) Now consider D_n/P_1 . The roots $\alpha \in \Delta(\mathfrak{g}_1)$ are of the form $\alpha = \alpha_1 + \dots + \alpha_i$, with $i \leq n$; $\alpha = \alpha_1 + \dots + \alpha_{n-2} + \alpha_n$ or $\alpha = \alpha_1 + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$, with $1 < j < n - 1$.

(II.A) Suppose $n - 1, n \in J$. The $B = (b_1, \dots, b_p)$ with $|B| = \mathfrak{a}$ are of the form $(1^{\mathfrak{a}}, 0^{\mathfrak{p}-\mathfrak{a}})$, if $\mathfrak{p} \geq \mathfrak{a} + 2$; $(1^{\mathfrak{a}}, 0)$ or $(1^{\mathfrak{a}-1}, 0, 1)$, if $\mathfrak{p} = \mathfrak{a} + 1$; $(1^{\mathfrak{a}-2m-2}, 2^m, 1, 1)$, if $\mathfrak{p} \leq \mathfrak{a}$. Note that $\mathfrak{a} + 1 \leq \mathfrak{p} \leq \mathfrak{a} + 2$, else $\mathfrak{g}_{-\alpha_n} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . If $j \in J - \{n - 1, n\}$, then $\mathfrak{g}_{-\alpha_j}$ stabilizes \mathfrak{n}_w . Thus, $J = \{n - 1, n\}$ and $\mathfrak{a} = 0, 1$.

If $\mathfrak{a} = 0$ then the lemma is immediate from the irreducibility (Lemma 3.11) of $\mathfrak{g}_{-1,0}$. If $\mathfrak{a} = 1$, then $\mathfrak{g}_{-1,-1} = \mathfrak{g}_{-1,(-1,0)} \oplus \mathfrak{g}_{-1,(0,-1)}$. Without loss of generality $\mathfrak{g}_{-1,(0,-1)} \subset \mathfrak{n}_w$. If $\mathfrak{g}_{-1,(-1,0)} \not\subset \mathfrak{n}_w$, then $\mathfrak{g}_{-\alpha_n} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w , a contradiction. Lemma 3.15 now follows, in this case, from Lemma 3.13.

(II.B) Suppose $n \in J$ and $n - 1 \notin J$. The $B = (b_1, \dots, b_p)$ with $|B| = \mathfrak{a}$ are of the form: $(1^{\mathfrak{a}}, 0^{\mathfrak{p}-\mathfrak{a}})$, if $\mathfrak{a} \leq \mathfrak{p}$; or $(1^{\mathfrak{a}-2m-1}, 2^m, 1)$, if $\mathfrak{p} \leq \mathfrak{a}$. Note that $\mathfrak{p} = \mathfrak{a} + 1$, else $\mathfrak{g}_{-\alpha_n} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . If $j \in J - \{n\}$, then $\mathfrak{g}_{-\alpha_j} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Thus $J = \{n\}$ and $\mathfrak{a} = 0$.

A similar argument will show that, if $n - 1 \in J$ and $n \notin J$, then $J = \{n - 1\}$ and $\mathfrak{a} = 0$. In both cases the lemma follows from the fact (Lemma 3.11) that $\mathfrak{g}_{-1,-\mathfrak{a}} = \mathfrak{g}_{-1,0}$ is irreducible.

(II.C) Suppose that $n - 1, n \notin J$. The $B = (b_1, \dots, b_p)$ with $|B| = \mathfrak{a}$ are of the form: $(1^{\mathfrak{a}}, 0^{\mathfrak{p}-\mathfrak{a}})$, if $\mathfrak{a} \leq \mathfrak{p}$; or $(1^{\mathfrak{a}-2m}, 2^m)$, if $\mathfrak{p} = \mathfrak{a} - m \leq \mathfrak{a}$. Note that $\mathfrak{a} \leq \mathfrak{p} \leq \mathfrak{a} + 1$, else $\mathfrak{g}_{-\alpha_{j\mathfrak{p}}} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . In either case, if $j \in J - \{j_{\mathfrak{p}}\}$, then $\mathfrak{g}_{-\alpha_j} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Thus $\mathfrak{p} = 1$, so that $J = \{j_1\}$, and $\mathfrak{a} = 0, 1$.

If $\mathfrak{a} = 0$, then the lemma follows from the irreducibility (Lemma 3.11) of $\mathfrak{g}_{-1,0}$. If $\mathfrak{a} = 1$, then the lemma follows from both Lemma 3.13 and the irreducibility of $\mathfrak{g}_{-1,0}$ and $\mathfrak{g}_{-1,-1}$.

(III) Next consider C_n/P_n . The roots $\alpha \in \Delta(\mathfrak{g}_1)$ are of the form $\alpha = \alpha_i + \dots + \alpha_n$, $\alpha = \alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n$ or $\alpha = 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n$. So the $B = (b_1, \dots, b_p)$ with $|B| = \mathfrak{a}$ are of the form $B_m = (0^{\mathfrak{p}-\mathfrak{a}+m}, 1^{\mathfrak{a}-2m}, 2^m)$, where $0 \leq 2m \leq \mathfrak{a}$. Arguing as above we note that $\mathfrak{p} \leq \mathfrak{a} + 1$, else the root space $\mathfrak{g}_{-\alpha_{j_1}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w . Similarly, $\mathfrak{a} \leq \mathfrak{p}$ else $\mathfrak{g}_{-\alpha_{j\mathfrak{p}}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w .

By definition of \mathbf{a} , there exists $A = B_{m(A)}$ such that $|A| = \mathbf{a}$ and $\mathfrak{g}_{-1,-A} \subset \mathfrak{n}_w$. Suppose that $\mathfrak{g}_{-1,-B_{m(A)+1}} \not\subset \mathfrak{n}_w$. Let $s = p - \ell(A) - m(A) + 1$ be the number of leading zeros in $B_{m(A)+1}$. Then, with some thought and Lemma 3.13, we see that the root space $\mathfrak{g}_{-\alpha_{j_s}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w , a contradiction. Arguing by induction, we conclude that $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for all $m \geq m(A)$.

For the other direction, suppose that $\mathfrak{g}_{-1,-B_{m(A)-1}} \not\subset \mathfrak{n}_w$. Let $s = p - m(A) + 1$ be the number of leading 0's and 1's in $B_{m(A)-1}$. Then, with some thought and Lemma 3.13, we see that the root space $\mathfrak{g}_{-\alpha_{j_s}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w . Arguing by induction, we conclude that $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for all $m \leq m(A)$. This establishes the lemma for C_n/P_n .

(IV) Next consider D_n/P_n . The roots of $\Delta(\mathfrak{g}_1)$ are of the form $\alpha = \alpha_i + \cdots + \alpha_n$, $\alpha = \alpha_i + \cdots + \alpha_{n-2} + \alpha_n$ and $\alpha = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$.

(IV.A) Suppose $n-1 \notin J$. The $B = (b_1, \dots, b_p)$ with $|B| = \mathbf{a}$ are of the form $B_m = (0^{p-a+m}, 1^{a-2m}, 2^m)$ with $0 \leq 2m \leq \mathbf{a}$. Note that $p \leq \mathbf{a} + 1$ else $\mathfrak{g}_{-\alpha_{j_1}} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Similarly, $p \geq \mathbf{a}$ else $\mathfrak{g}_{-\alpha_{j_p}} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Thus, $\mathbf{a} \leq p \leq \mathbf{a} + 1$.

By definition of A , there exists an $A = B_{m(A)}$ such that $\mathfrak{g}_{-1,-A} \subset \mathfrak{n}_w$. Suppose that $\mathfrak{g}_{-1,-B_{m(A)+1}} \not\subset \mathfrak{n}_w$. Let $s = p - \mathbf{a} + m(A) + 1$ be the number of leading zeros in $B_{m(A)+1}$. Then $\mathfrak{g}_{-\alpha_{j_s}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w , a contradiction. Arguing by induction, we conclude that $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for all $m \geq m(A)$.

In the other direction, suppose that $\mathfrak{g}_{-1,-B_{m(A)-1}} \not\subset \mathfrak{n}_w$. Let $s = p - m(A) + 1$ be the number of leading 0's and 1's in $B_{m(A)-1}$. Then, with some thought, we see that the root space $\mathfrak{g}_{-\alpha_{j_s}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w . Arguing by induction, we conclude that $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for all $m \leq m(A)$. This establishes the lemma for $n-1 \notin J$. The following observations will be helpful in Section 6.4.

Remark 3.16a. Assume that $n-1 \notin J$. If $\mathbf{a} = 1$, then $J \neq \{1\}$ (else $\mathfrak{n}_w = \mathfrak{g}_{-1}$ and $X_w = G/P$).

Consider the tuple $B = (0^{p-r}, 2^r)$ with $r \geq 1$, which occurs when $\mathbf{a} = 2r \geq 2$. This tuple will only appear if $1 < j_{p-r+1} - j_{p-r}$. If it does not appear, then $\mathfrak{g}_{-\alpha_{j_{p-r}}}$ will stabilize \mathfrak{n}_w , a contradiction. We may conclude that $1 < j_{p-r+1} - j_{p-r}$.

Consider the tuple $B = (0^{p-r}, 1, 2^{r-1})$ with $r \geq 1$, which occurs when $\mathbf{a} = 2r - 1 \geq 1$. Note that $\mathfrak{g}_{-\alpha_{j_{p-r+1}}}$ will stabilize \mathfrak{n}_w if $1 = j_{p-r+1} - j_{p-r}$. Therefore $1 < j_{p-r+1} - j_{p-r}$.

(IV.B) Suppose $n-1 \in J$. The $B = (b_1, \dots, b_p)$ with $|B| = \mathbf{a}$ are of the form $B^0 = (0^{p-a-1}, 1^a, 0)$, or $B_m = (0^{p-a+m}, 1^{a-2m-1}, 2^m, 1)$ with $0 \leq 2m \leq \mathbf{a} - 1$. Note that $p \leq \mathbf{a} + 2$, else $\mathfrak{g}_{-\alpha_{j_1}} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w . Similarly $\mathbf{a} + 1 \leq p$, else $\mathfrak{g}_{-\alpha_{n-1}} \subset \mathfrak{g}_{0,-1}$ stabilizes \mathfrak{n}_w .

An argument similar to that for $n-1 \notin J$ implies if $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for some m , then $\mathfrak{g}_{-1,-B_m} \subset \mathfrak{n}_w$ for every m . We leave this to the reader.

Assume that no $B_{-1,B_m} \subset \mathfrak{n}_w$. Then by definition of \mathbf{a} the irreducible $\mathfrak{g}_{-1,-B^0}$ must lie in \mathfrak{n}_w . With some thought we see that $\mathfrak{g}_{-\alpha_{j_{p-a}}} \subset \mathfrak{g}_{0,-}$ will stabilize \mathfrak{n}_w , a contradiction. So we may conclude that every $\mathfrak{g}_{-1,-B_m}$ lies in \mathfrak{n}_w . It remains to show that $\mathfrak{g}_{-1,-B^0}$ also lies in \mathfrak{n}_w . But if it does not, then $\mathfrak{g}_{-\alpha_{j_1}} \subset \mathfrak{g}_{0,-}$ stabilizes \mathfrak{n}_w , a contradiction. This establishes the lemma for D_n/P_n . The following observations will be helpful in Section 6.4.

Remark 3.16b. Assume $n-1 \in J$. If $\mathbf{a} = 0$, then $n-2 \notin J$ (else $\mathfrak{g}_{-\alpha_{n-1}}$ stabilizes \mathfrak{n}_w , contradicting (3.5)). If $\mathbf{a} = 1$, then $n-2 \notin J$ (else $\mathfrak{g}_{-\alpha_{n-2}}$ stabilizes \mathfrak{n}_w).

Consider the tuple $B = (0^{p-r-1}, 2^r, 1)$ with $r \geq 1$, which occurs when $a = 2r + 1 \geq 3$. This tuple will appear only if $1 < j_{p-r} - j_{p-r-1}$. If the tuple does not appear, $\mathfrak{g}_{-\alpha_{j_{p-r-1}}}$ will stabilize \mathfrak{n}_w . Thus, $1 < j_{p-r} - j_{p-r-1}$.

Consider the tuple $B = (0^{p-r-1}, 1, 2^{r-1}, 1)$ with $r \geq 1$, which occurs when $a = 2r \geq 2$. Note that $\mathfrak{g}_{-\alpha_{j_{p-r}}}$ will stabilize \mathfrak{n}_w if $1 = j_{p-r} - j_{p-r-1}$. Therefore, $1 < j_{p-r} - j_{p-r-1}$.

(V) We finish with A_n/P_i , for any $1 \leq i \leq n$. Any root $\alpha \in \Delta(\mathfrak{g}_1)$ is of the form $\alpha = \alpha_i + \dots + \alpha_j$ with $i \leq i \leq j$. In particular, the $B = (b_1, \dots, b_p)$ with $|B| = a$ are of the form $B_\ell = (0^\ell, 1^a, 0^{p-a-\ell})$, with $0 \leq \ell \leq p - a$. In particular, $a \leq p$. If $a = p$, then there is a unique B with $|B| = a$ and $\mathfrak{n}_w = \mathfrak{g}_{-1}$. In particular, $X_w = X$ is not a proper subvariety.

So $a < p$. By definition of a , there exists $A = B_{\ell(A)}$ such that $|A| = a$ and $\mathfrak{g}_{-1, -A} \subset \mathfrak{n}_w$. Suppose that $\mathfrak{g}_{-1, -B_{\ell(A)+1}} \not\subset \mathfrak{n}_w$. Then with some thought we see that the root space $\mathfrak{g}_{-\alpha_{j_{\ell(A)+1}}} \subset \mathfrak{g}_{0, -}$ stabilizes \mathfrak{n}_w , a contradiction. By induction, it follows that $\mathfrak{g}_{-1, -B_\ell} \subset \mathfrak{n}_w$ for all $\ell \geq \ell(A)$.

Next suppose that $\mathfrak{g}_{-1, -B_{\ell(A)-1}} \not\subset \mathfrak{n}_w$. Again, with some thought, we see that the root space $\mathfrak{g}_{-\alpha_{j_{\ell(A)+a}}} \subset \mathfrak{g}_{0, -}$ stabilizes, a contradiction. We conclude that $\mathfrak{g}_{-1, -B_\ell} \subset \mathfrak{n}_w$ for all $\ell \leq \ell(A)$.

Given $J \neq \emptyset$, suppose that $i < j_1$. Observe that $\mathfrak{g}_{-\alpha_{j_p}}$ will stabilize \mathfrak{n}_w if $p > a + 1$. Thus $|J| = a + 1$. Moreover, if $j \in J - \{j_p\}$, then $\mathfrak{g}_{-\alpha_j} \subset \mathfrak{g}_{0, -1}$ stabilizes \mathfrak{n}_w . Thus $|J| = 1$ and $a = 0$. Similarly, if $j_p < i$, then $|J| = 1$ and $a = 0$.

Suppose that $j_1 < i < j_p$. Define $q \in \mathbb{Z}$ by $j_q < i < j_{q+1}$. Observe that $q \leq a + 1$, else $\mathfrak{g}_{-\alpha_{j_1}}$ will stabilize \mathfrak{n}_w . Similarly, $p - q \leq a + 1$, else $\mathfrak{g}_{-\alpha_{j_p}}$ will stabilize \mathfrak{n}_w . Also $a \leq q$, else $\mathfrak{g}_{-\alpha_{j_{q+1}}}$ stabilizes \mathfrak{n}_w . Similarly, $a \leq p - q$, else $\mathfrak{g}_{-\alpha_{j_q}}$ stabilizes \mathfrak{n}_w . \square

The analyses of Sections 6 and 8 require a detailed description of (a, J) . To that end, the observations made in the proof of Lemma 3.15 are collected in the corollary below. For convenience we set

$$j_0 := 0 \quad \text{and} \quad j_{p+1} := 1 + \max\{1, \dots, \hat{i}, \dots, n\}.$$

so that $j_0 < j_1 < \dots < j_p < j_{p+1}$.

Corollary 3.17. *Let G/P_i be a classical compact Hermitian symmetric space. Suppose that $X_w \subset X$ is a proper Schubert variety. Then $0 \leq a(w) \in \mathbb{Z}$ and $\emptyset \neq J(w) = \{j_1, \dots, j_p\} \subset I_p$ satisfy the criteria in Table 2. Conversely, any (a, J) meeting the criteria in Table 2 is realized by a unique Schubert variety $X_w \subset X$; that is, there exists a unique $w \in W^p$ such that $a = a(w)$ and $J = J(w)$.*

Remark. Schubert varieties in the Grassmannian $\text{Gr}(i, n+1) = A_n/P_i$ are indexed by partitions $\pi \in P(i, n+1)$. Proposition 3.30 describes (a, J) in terms of the partition.

We close this section with some definitions. Given a Schubert variety X_w , from this point on $\mathfrak{g} = \oplus \mathfrak{g}_{j,k}$ will denote the (Z_i, Z_w) -bigraded decomposition of \mathfrak{g} . Define

$$(3.18) \quad \begin{aligned} \mathfrak{g}_w &:= \mathfrak{n}_w \oplus \mathfrak{g}_{0, \geq 0} \oplus \mathfrak{g}_{1, \geq a}, & \mathfrak{n}_w &= \mathfrak{g}_{-1, \geq -a}; \\ \mathfrak{g}_w^\perp &:= \mathfrak{n}_w^\perp \oplus \mathfrak{g}_{0, < 0} \oplus \mathfrak{g}_{1, < a}, & \mathfrak{n}_w^\perp &= \mathfrak{g}_{-1, < -a}. \end{aligned}$$

Both \mathfrak{g}_w and \mathfrak{g}_w^\perp are subalgebras of \mathfrak{g} , and \mathfrak{g}_w is the largest subalgebra of \mathfrak{g} containing \mathfrak{n}_w . Both $\mathfrak{g} = \mathfrak{g}_w \oplus \mathfrak{g}_w^\perp$ and $\mathfrak{g}_{-1} = \mathfrak{n}_w \oplus \mathfrak{n}_w^\perp$ are $\mathfrak{g}_{0,0}$ -module decompositions.

TABLE 2. (\mathbf{a}, \mathbf{J}) for the classical G/P_i

G/P_i	Upper bound on \mathbf{a}	Realizability criteria for (\mathbf{a}, \mathbf{J})	
A_n/P_i	$\min(i-1, n-i)$	Defining $0 \leq \mathbf{q} \leq \mathbf{p}$ by $j_{\mathbf{q}} < i < j_{\mathbf{q}+1}$, we have: $(\mathbf{p}, \mathbf{q}) \in \{(2\mathbf{a}, \mathbf{a}), (2\mathbf{a}+1, \mathbf{a}), (2\mathbf{a}+1, \mathbf{a}+1), (2\mathbf{a}+2, \mathbf{a}+1)\}$.	
B_n/P_1	1	$\mathbf{J} = \{\mathbf{j}\}$	
D_n/P_1	1	$\mathbf{a} = 0$, $\mathbf{J} = \{\mathbf{j}\}$ or $\{n-1, n\}$; $\mathbf{a} = 1$, $\mathbf{J} = \{\mathbf{j} \leq n-2\}$ or $\{n-1, n\}$.	
C_n/P_n	$n-1$	$\mathbf{p} = \mathbf{a}$ or $\mathbf{a}+1$.	
D_n/P_n	$n-3$	$\mathbf{p} = \mathbf{a}$ and $n-1 \notin \mathbf{J}$, or $\mathbf{p} = \mathbf{a}+1$ and $n-1 \in \mathbf{J}$	$j_s - j_{s-1} \geq 2$ for $s = \left\lceil \frac{\mathbf{p}+1}{2} \right\rceil$;
	$n-4$	$\mathbf{p} = \mathbf{a}+1$ and $n-1 \notin \mathbf{J}$, or $\mathbf{p} = \mathbf{a}+2$ and $n-1 \in \mathbf{J}$	$j_{s+1} - j_s \geq 2$ for $s = \left\lceil \frac{\mathbf{p}}{2} \right\rceil$.

3.3. Smooth Schubert varieties.

Proposition 3.19. *Let X be a compact Hermitian symmetric space. The Schubert variety $X_w \subset X$ is smooth if and only if the integer $\mathbf{a}(w)$ of Proposition 3.9 is zero.*

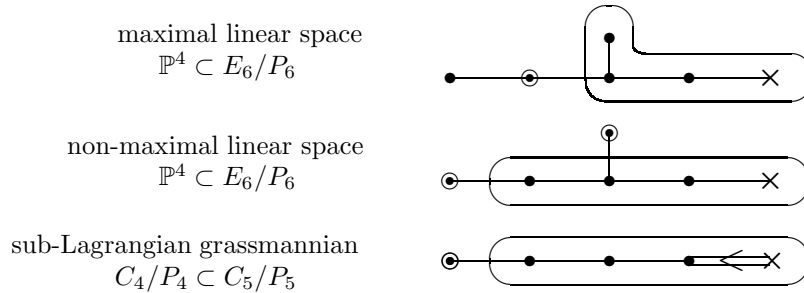
The proposition follows almost immediately from Proposition 3.9 and the following.

Proposition 3.20 ([10, §2.3]). *Let G/P_i be an irreducible compact Hermitian symmetric space. The Schubert variety X_w is smooth if and only if X_w is homogeneous. The smooth Schubert varieties (modulo the action of G) are in bijective correspondence with the connected sub-diagrams δ of the Dynkin diagram $\delta_{\mathfrak{g}}$ of G that contain the node \mathbf{i} . In particular, if $D \subset G$ is the simple Lie group associated to $\delta \subset \delta_{\mathfrak{g}}$, then $X_w = D/(D \cap P)$, and $\mathbf{n}_w = \mathfrak{d} \cap \mathfrak{g}_{-1}$.*

Proof of Proposition 3.19. Assume X_w is smooth and let δ be the associated sub-diagram. Let \mathbf{J} be given by the nodes of $\delta_{\mathfrak{g}} \setminus \delta$ that are adjacent to δ . Then the sub-diagram δ clearly corresponds to the \mathbf{n}_w associated to \mathbf{J} with $\mathbf{a} = 0$.

Conversely, given a Schubert variety X_w with associated \mathbf{J} and $\mathbf{a} = 0$, let δ be the largest connected sub-diagram of $\delta_{\mathfrak{g}}$ containing the node \mathbf{i} and with the property that no node of \mathbf{J} is contained in δ . If $\mathfrak{d} \subset \mathfrak{g}$ is the simple subalgebra associated to δ , then we clearly have $\mathbf{n}_w = \mathfrak{d} \cap \mathfrak{g}_{-1}$. \square

Example. The following examples illustrate the relationship between the sub-diagram δ of Proposition 3.20 and $\mathbf{J}(w)$. The \mathbf{i} -th node is marked with a \times , and nodes of \mathbf{J} are circled. These examples all have $\mathbf{a} = 0$.



3.4. Conjugation and duality in the CHSS. Fix $w \in W^{\mathfrak{p}}$ with associated $\mathbf{a} = \mathbf{a}(w)$ and $\mathbf{J} = \mathbf{J}(w)$. Let w' denote the conjugate, defined in Section 2.5 with respect to a choice of Dynkin diagram automorphism $\varphi : \delta_{\mathfrak{g}} \rightarrow \delta_{\mathfrak{g}}$. It is straight-forward to check that

$$(3.21) \quad \mathbf{a}' := \mathbf{a}(w') = \mathbf{a} \quad \text{and} \quad \mathbf{J}' := \mathbf{J}(w') = \varphi(\mathbf{J}).$$

By Corollary 7.2, X_{w^*} is Poincaré dual to X_w . The rest of this section is devoted to the determination of $\mathbf{a}(w^*)$ and $\mathbf{J}(w^*)$. Given $*$: $I_{\mathfrak{p}} \rightarrow I_{\mathfrak{p}}$ from Section 2.5 which induces (2.9), let $\mathbf{J}^* := *\mathbf{J}$. As noted in Remark 2.10, there is an induced automorphism ψ of the Dynkin sub-diagram $\delta_{\mathfrak{p}} \subset \delta_{\mathfrak{g}}$. By Lemma 3.22(a), identifying ψ will determine $\mathbf{J}(w^*)$.

Lemma 3.22. *Let $G/P_{\mathbf{i}}$ be CHSS. Let $w \in W^{\mathfrak{p}}$ with $(\mathbf{a}, \mathbf{J}) = (\mathbf{a}(w), \mathbf{J}(w))$. Then*

- (a) $\mathbf{J}(w^*) = \mathbf{J}^*$;
- (b) $w_{\mathfrak{p}}^0 \alpha_{\mathbf{i}}$ is the highest root $\tilde{\alpha} \in \Delta(\mathfrak{g}_1)$ of \mathfrak{g} ;
- (c) $\mathbf{a}^* := \mathbf{a}(w^*) = \tilde{\alpha}(Z_w) - \mathbf{a} - 1$.

Proof. (a) is deduced from Lemma 2.8 as follows:

$$\begin{aligned} \mathbf{j} \in \mathbf{J} & \stackrel{(3.5)}{\iff} \exists \beta \in \Delta(w) \text{ with } \beta + \alpha_{\mathbf{j}} \in \Delta(\mathfrak{g}_1) \setminus \Delta(w) \\ & \iff \exists \gamma \in \Delta(w^*) \text{ with } \gamma - w_{\mathfrak{p}}^0 \alpha_{\mathbf{j}} \in \Delta(\mathfrak{g}_1) \setminus \Delta(w^*) \quad [\text{Set } \gamma = w_{\mathfrak{p}}^0(\beta + \alpha_{\mathbf{j}}).] \\ & \stackrel{(3.5)}{\iff} \mathbf{j}^* \in \mathbf{J}(w^*). \end{aligned}$$

For (b), we note from (2.7) that $w_{\mathfrak{p}}^0 \alpha_{\mathbf{i}} \in \Delta(\mathfrak{g}_1)$, so it suffices to show that $w_{\mathfrak{p}}^0(\alpha_{\mathbf{i}}) + \alpha_{\mathbf{j}} \notin \Delta$ for all simple roots $\alpha_{\mathbf{j}}$. The equation (3.2) implies $w_{\mathfrak{p}}^0 \alpha_{\mathbf{i}} + \alpha_{\mathbf{i}} \notin \Delta$. Let $j \in I_{\mathfrak{p}}$. If $w_{\mathfrak{p}}^0 \alpha_{\mathbf{i}} + \alpha_{\mathbf{j}} \in \Delta(\mathfrak{g}_1)$, then $\alpha_{\mathbf{i}} - \alpha_{\mathbf{j}^*} = \alpha_{\mathbf{i}} + w_{\mathfrak{p}}^0 \alpha_{\mathbf{j}} \in \Delta(\mathfrak{g}_1)$, a contradiction.

To prove (c), note first that by Proposition 3.9, $\Delta(\mathfrak{g}_1) \setminus \Delta(w) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \alpha(Z_w) > \mathbf{a}\}$. Given $\alpha \in \Delta(\mathfrak{g}_1)$, write $\alpha = \alpha_{\mathbf{i}} + \sum_{j \in I_{\mathfrak{p}}} m^j \alpha_{\mathbf{j}}$, where $m^j \in \mathbb{Z}_{\geq 0}$. Then $\alpha \in \Delta(\mathfrak{g}_1) \setminus \Delta(w)$ if and only if $\sum_{j \in \mathbf{J}} m^j > \mathbf{a}$. By Lemma 2.8(c), every $\alpha^* \in \Delta(w^*)$ is of the form $w_{\mathfrak{p}}^0(\alpha)$ for some $\alpha \in \Delta(\mathfrak{g}_1) \setminus \Delta(w)$. From Lemma 3.22(b), we deduce

$$\alpha^*(Z_{w^*}) = w_{\mathfrak{p}}^0(\alpha)(Z_{w^*}) = \tilde{\alpha}(Z_{w^*}) - \sum_{j \in \mathbf{J}} m^j < \tilde{\alpha}(Z_{w^*}) - \mathbf{a}.$$

Thus, $\mathbf{a}^* = \tilde{\alpha}(Z_{w^*}) - \mathbf{a} - 1$. It is straight-forward to check that $\tilde{\alpha}(Z_{w^*}) = \tilde{\alpha}(Z_w)$. \square

Remark 3.23. If G/P is a classical CHSS, then the values of $\tilde{\alpha}(Z_w)$ are as follows.

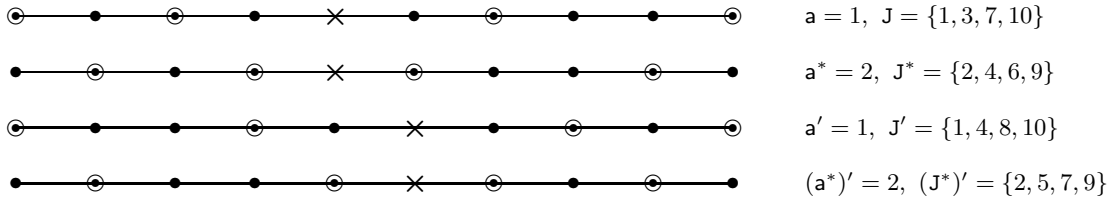
- (a) for $A_n/P_{\mathbf{i}}$, we have $\tilde{\alpha}(Z_w) = |\mathbf{J}(w)|$;
- (b) for B_n/P_1 and C_n/P_n , we have $\tilde{\alpha}(Z_w) = 2|\mathbf{J}(w)|$;
- (c) for $D_n/P_{\mathbf{i}}$, with $\mathbf{i} = 1, n-1, n$ and $\mathbf{I} = \{1, n-1, n\} \setminus \{\mathbf{i}\}$, we have
 - (i) $\tilde{\alpha}(Z_w) = 2|\mathbf{J}(w)|$, if $\mathbf{I} \cap \mathbf{J}(w) = \emptyset$,
 - (ii) $\tilde{\alpha}(Z_w) = 2|\mathbf{J}(w)| - 2$, if $\mathbf{I} \subset \mathbf{J}(w)$, and
 - (iii) $\tilde{\alpha}(Z_w) = 2|\mathbf{J}(w)| - 1$ otherwise.

Proposition 3.24. *Let $G/P_{\mathbf{i}}$ be an irreducible compact Hermitian symmetric space. The automorphism ψ of the Dynkin sub-diagram $\delta_{\mathfrak{p}} = \delta_{\mathfrak{g}} \setminus \{\mathbf{i}\}$ corresponding to $*$: $I_{\mathfrak{p}} \rightarrow I_{\mathfrak{p}}$ preserves each connected component $\delta_0 \subset \delta_{\mathfrak{p}}$. In the case D_n/P_1 , with n odd, ψ acts trivially. In all other cases ψ acts on δ_0 by the unique nontrivial automorphism, when it exists.*

Remark. The sub-diagram $\delta_{\mathfrak{p}}$ is connected for all CHSS, except $A_n/P_{\mathfrak{i}}$, with $1 < \mathfrak{i} < n$.

Proof. The first half of the proposition follows from Lemma 3.22 and the following well-known results from representation theory. Let $\delta_0 \subset \delta_{\mathfrak{p}}$ be a connected component. Since $w_{\mathfrak{p}}^0$ preserves the root subsystem associated to δ_0 , it is immediate from the definition (2.9) that ψ preserves δ_0 . If δ_0 is of type \mathfrak{a}_{ℓ} , then $w_{\mathfrak{p}}^0$ acts on δ_0 by the unique nontrivial (if $\ell > 1$) automorphism. If δ_0 is of type \mathfrak{d}_{ℓ} , then $w_{\mathfrak{p}}^0$ acts on δ_0 by the unique nontrivial automorphism if ℓ is odd; and trivially if ℓ is even. If δ_0 is of type \mathfrak{e}_6 , then $w_{\mathfrak{p}}^0$ acts on δ_0 by the unique nontrivial automorphism. In all other cases δ_0 admits no nontrivial automorphism. \square

Example 3.25. In A_{10}/P_5 , consider $\mathfrak{a} = 1$ and $J = \{1, 3, 7, 10\}$ which corresponds by Corollary 3.17 to some X_w . By Remark 3.23(a), $\tilde{\alpha}(Z_J) = 4$. Using (3.21), Lemma 3.22, and Proposition 3.24, we obtain:



3.5. Schubert varieties in Grassmannians. It is well-known that Schubert varieties in $\text{Gr}(\mathfrak{i}, n+1) = A_n/P_{\mathfrak{i}}$ are indexed by partitions

$$(3.26) \quad \mathcal{P}(\mathfrak{i}, n+1) = \{(a_1, \dots, a_{\mathfrak{i}}) \in \mathbb{Z}^{\mathfrak{i}} : n+1-\mathfrak{i} \geq a_1 \geq a_2 \geq \dots \geq a_{\mathfrak{i}} \geq 0\}.$$

Thus, the Hasse diagram $W^{\mathfrak{p}}$ and $\mathcal{P}(\mathfrak{i}, n+1)$ are in one-to-one correspondence, and we may label a Schubert variety as X_{π} for some $\pi \in \mathcal{P}(\mathfrak{i}, n+1)$. In this section we derive the formulas for $\mathfrak{a} = \mathfrak{a}(\pi)$ and $J = J(\pi)$. This will provide a dictionary between the familiar partition description of the Schubert varieties, and our (\mathfrak{a}, J) -description.

An element $\pi = (a_1, \dots, a_{\mathfrak{i}}) \in \mathcal{P}(\mathfrak{i}, n+1)$ is a Young diagram with a_{ℓ} boxes in row ℓ , drawn on an $\mathfrak{i} \times (n+1-\mathfrak{i})$ rectangular grid. We define $|\pi| = a_1 + \dots + a_{\mathfrak{i}}$. As π may contain repetitions, write $\pi = (p_1^{q_1}, \dots, p_r^{q_r})$, where $\{p_{\ell}\}_{\ell=1}^r$ is strictly decreasing, $p_{\ell}, q_{\ell} \geq 1$ for all ℓ , and

$$(3.27) \quad p_1 \leq n+1-\mathfrak{i} \quad \text{and} \quad q_1 + \dots + q_r \leq \mathfrak{i}.$$

The first q_1 rows of π each have p_1 boxes, et cetera. (The $\mathfrak{i} - \sum_{j=1}^r q_j$ zeros have been suppressed.)

We will consider two operations on partitions: conjugation $' : \mathcal{P}(\mathfrak{i}, n+1) \rightarrow \mathcal{P}(n+1-\mathfrak{i}, n+1)$ and duality $* : \mathcal{P}(\mathfrak{i}, n+1) \rightarrow \mathcal{P}(\mathfrak{i}, n+1)$. Given $\pi = (p_1^{q_1}, \dots, p_r^{q_r}) \in \mathcal{P}(\mathfrak{i}, n+1)$,

- (1) π' is the transposed Young diagram, and
- (2) π^* is the complement within the $\mathfrak{i} \times (n+1-\mathfrak{i})$ rectangular grid, rotated 180 degrees.

It is clear that $(\pi')' = \pi = (\pi^*)^*$, $(\pi^*)' = (\pi')^*$, $|\pi'| = |\pi|$, and $|\pi| + |\pi^*| = \mathfrak{i}(n+1-\mathfrak{i})$.

If $\pi' = (p_1'^{q_1'}, \dots, p_{r'}'^{q_{r'}'})$, then $r' = r$ and

$$(3.28) \quad p_i' = q_1 + \dots + q_{r-i+1} \quad \text{and} \quad q_i' = p_{r-i+1} - p_{r-i+2}$$

for all $i = 1, \dots, r$ and $p_{r+1} := 0$. Note that $p'_1 = q_1 + \dots + q_r$. We decompose the partitions into four types.

Definition 3.29. We decompose $P(i, n+1)$ in to four types:

$$\begin{aligned} \pi \in \spadesuit & \text{ if } p_1 = n+1-i \text{ and } p'_1 = i; & \pi \in \heartsuit & \text{ if } p_1 = n+1-i \text{ and } p'_1 < i; \\ \pi \in \diamond & \text{ if } p_1 < n+1-i \text{ and } p'_1 = i; & \pi \in \clubsuit & \text{ if } p_1 < n+1-i \text{ and } p'_1 < i. \end{aligned}$$

Remark. Note that $p_1 = n+1-i$ [respectively, $p'_1 = i$] precisely when the first row [respectively, column] of π achieves the maximum possible length. Hence,

$$\begin{aligned} \text{Conjugation: } \spadesuit' &= \spadesuit, \quad \clubsuit' = \clubsuit, \quad \heartsuit' = \diamond. \\ \text{Duality: } \spadesuit^* &= \clubsuit, \quad \heartsuit^* = \heartsuit, \quad \diamond^* = \diamond. \end{aligned}$$

$$\text{The partition } \pi^* = (p_1^{*q_1^*}, \dots, p_r^{*q_r^*}) \text{ is given by } r^* = \begin{cases} r-1, & \text{if } \pi \in \spadesuit, \\ r, & \text{if } \pi \in \heartsuit \cup \diamond, \\ r+1, & \text{if } \pi \in \clubsuit; \end{cases} \text{ and}$$

$$\begin{aligned} (p_1^*, q_1^*) &= \begin{cases} (n+1-i, i-p'_1), & \text{if } \pi \in \heartsuit \cup \clubsuit, \\ (n+1-i-p_r, q_r), & \text{if } \pi \in \spadesuit \cup \diamond, \end{cases} \\ (p_\ell^*, q_\ell^*) &= \begin{cases} (n+1-i-p_{r-\ell+2}, q_{r-\ell+2}), & \text{if } \pi \in \heartsuit \cup \clubsuit, \\ (n+1-i-p_{r-\ell+1}, q_{r-\ell+1}), & \text{if } \pi \in \spadesuit \cup \diamond, \end{cases} \quad \text{where } 1 < \ell < r, \\ (p_r^*, q_r^*) &= \begin{cases} (n+1-i-p_2, q_2), & \text{if } \pi \in \heartsuit \cup \clubsuit, \\ (n+1-i-p_1, q_1), & \text{if } \pi \in \diamond, \end{cases} \\ (p_{r+1}^*, q_{r+1}^*) &= (n+1-i-p_1, q_1), \quad \text{if } \pi \in \clubsuit. \end{aligned}$$

Proposition 3.30. *Given $\pi = (p_1^{q_1}, \dots, p_r^{q_r}) \in P(i, n+1)$, the Schubert variety $X_\pi \subset A_n/P_i$ has $\dim(X_\pi) = |\pi^*|$ and*

$$\begin{aligned} a(\pi) &= r^* - 1 = \text{the number of interior corners of } \pi^*, \\ J(\pi) &= \{q_1, q_1 + q_2, \dots, q_1 + \dots + q_r, n+1-p_1, n+1-p_2, \dots, n+1-p_r\} \setminus \{i\}. \end{aligned}$$

Remark. By (3.27), the $J(\pi)$ in Proposition 3.30 is a strictly increasing sequence.

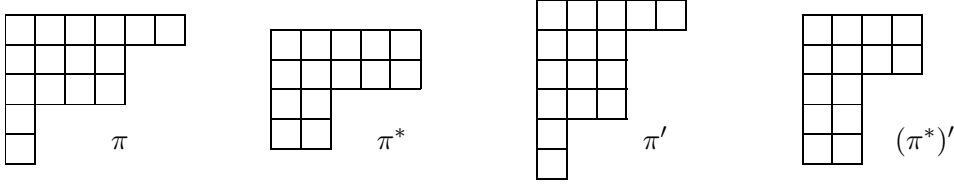
Example. The example $X_\pi = \sigma(W) \subset \text{Gr}(m, n) = A_{n-1}/P_m$ discussed in Section 1.1 corresponds to the partition $\pi = (k)$; that is, $(p_1, q_1) = (k, 1)$. If $k = n$, then $\pi \in \spadesuit$, $a = 0$ and $J = \{1\}$; if $1 \leq k < n$ then $\pi \in \clubsuit$, $a = 1$ and $J = \{1, n-k\} \setminus \{m\}$.

There are two immediate corollaries to Proposition 3.30. The first is a second proof of Corollary 3.17, with the correspondences of Table 3. The second corollary is the observation that the conjugate and duality notations of Sections 2.5 & 3.5 are consistent: if $w \in W^p$ corresponds to $\pi \in P(i, n+1)$, then w^* corresponds to π^* and w' to π' .

TABLE 3. Suits versus (p, q) .

Type	\spadesuit	\heartsuit	\diamond	\clubsuit
(p, q)	$(2a+2, a+1)$	$(2a+1, a+1)$	$(2a+1, a)$	$(2a, a)$

Example 3.31. By Proposition 3.30, the Schubert variety of Example 3.25 with $a = 1$, $J = \{1, 3, 7, 10\}$ corresponds to $\pi = (6, 4^2, 1^2) \in P(5, 11)$, sitting in a 5×6 rectangle.



We have $\pi, \pi' \in \spadesuit$, while $\pi^*, (\pi^*)' \in \clubsuit$.

Proof of Proposition 3.30. The stabilizer $P = P_{\mathbf{i}} \subset A_n = SL_{n+1}(\mathbb{C})$ has block form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the diagonal square blocks are of sizes \mathbf{i} and $n+1-\mathbf{i}$ respectively. Hence, $T_o(A_n/P_{\mathbf{i}}) \cong \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = \mathfrak{g}_{-1}$ and $T_o X_\pi$ is identified with \mathfrak{n}_π . Here, $\mathfrak{n}_\pi \subset \text{Mat}_{n+1-\mathbf{i}, \mathbf{i}}(\mathbb{C})$ is the matrix subspace such that $(\mathfrak{n}_\pi)_{ij} = 0$ if $i > n+1-\mathbf{i}-a_j$ and we embed this as $\begin{pmatrix} 0 & 0 \\ \mathfrak{n}_\pi & 0 \end{pmatrix} \subset \mathfrak{g}_{-1}$, cf. [3, §2.2]. Pictorially, the nonzero entries of \mathfrak{n}_π are precisely the entries of π^* rotated clockwise by 90 degrees. Consequently, $\dim(X_\pi) = |\pi^*| = (n+1-\mathbf{i})\mathbf{i} - |\pi|$.

Note that \mathfrak{g}_0 is block diagonal with blocks of size \mathbf{i} and $n+1-\mathbf{i}$. Recall that \mathbf{J} is determined by the parabolic subalgebra in \mathfrak{g}_0 stabilizing \mathfrak{n}_π . We have

$$\left[\begin{pmatrix} K & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \mathfrak{n}_\pi & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ L\mathfrak{n}_\pi - \mathfrak{n}_\pi K & 0 \end{pmatrix}.$$

So $\mathfrak{g}_{0,0} = \{(K, L) \mid L\zeta - \zeta K \in \mathfrak{n}_\pi \ \forall \ \zeta \in \mathfrak{n}_\pi\}$. Before identifying $\mathfrak{g}_{0,0}$, we pause to consider an example.

Example 3.32. It will be useful to draw a grid on $\text{Mat}_{n+1-\mathbf{i}, \mathbf{i}}(\mathbb{C})$: at every corner in \mathfrak{n}_π , draw a horizontal and vertical line. This yields a partition of the row and column spaces $\text{Mat}_{n+1-\mathbf{i}, \mathbf{i}}(\mathbb{C})$. In turn, these respectively induce natural partitions of $\text{Mat}_{n+1-\mathbf{i}, n+1-\mathbf{i}}(\mathbb{C})$ and $\text{Mat}_{\mathbf{i}, \mathbf{i}}(\mathbb{C})$. For $\pi \in \mathcal{P}(5, 11)$ as in Example 3.31, $\pi^* = (5^2, 2^2)$ and the induced grid is

$$\mathfrak{n}_\pi = \left(\begin{array}{c|cc|cc} 0 & * & * & * & * \\ 0 & * & * & * & * \\ \hline 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right) \sim \begin{pmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix},$$

where the final matrix above is a block simplification of \mathfrak{n}_π . The starred entries in \mathfrak{n}_π above are *arbitrary*, so to satisfy $L\mathfrak{n}_\pi - \mathfrak{n}_\pi K \subset \mathfrak{n}_\pi$, both K and L are of the form $\begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$,

but the diagonal block sizes for each are different. For K , they are sizes 1, 2, 2, while for L , they are sizes 2, 3, 1. The parabolic $\mathfrak{g}_{0, \geq 0} \subset \mathfrak{g}_0$ determined by $\text{diag}(K, L)$ has associated index set $\mathbf{J} = \{1, 3, 7, 10\}$. (Note that we do not include 5 since $\mathbf{i} = 5$ here.) The reductive subalgebra $\mathfrak{g}_{0,0}$ is block diagonal with blocks of sizes 1, 2, 2, 2, 3, 1. In the last column of \mathfrak{n}_π , we have from top to bottom the eigenspaces corresponding to the roots

$$-\alpha_5, \quad -\alpha_5 - \alpha_6, \quad -\alpha_5 - \alpha_6 - \alpha_7, \quad -\alpha_5 - \cdots - \alpha_8, \quad -\alpha_5 - \cdots - \alpha_9.$$

These roots all have $Z_{\mathbf{i}}$ -grading -1 , and their respective Z_J -gradings are $0, 0, -1, -1, -1$. In the block simplification of \mathbf{n}_π , the Z_J -gradings are given below, with \mathbf{n}_π also indicated:

$$\begin{pmatrix} -2 & \boxed{-1} & 0 \\ -3 & -2 & \boxed{-1} \\ -4 & -3 & -2 \end{pmatrix}$$

Thus, $\mathbf{n}_\pi = \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{-1,-1}$, so $\mathbf{a} = 1$ and this is the same as the number of interior corners of π^* . Finally, $[\mathfrak{g}_{0,0}, \mathbf{n}_\pi] \subset \mathbf{n}_\pi$, so $\mathfrak{g}_{0,0}$ preserves \mathbf{n}_π . Moreover, $\mathfrak{g}_{0,0}$ acts *irreducibly* on every block of \mathbf{n}_π . Hence, $\mathfrak{g}_{-1,0}$ and $\mathfrak{g}_{-1,-1}$ have one and two $\mathfrak{g}_{0,0}$ -irreducible components of dimension 4 and 4, 6 respectively.

We now return to the proof of the proposition. The argument proceeds exactly as in the above example. There are four types of π^* to consider (Definition 3.29). In all cases, \mathbf{J} is found to be

$$\mathbf{J} = \left\{ \mathbf{i} - \sum_{\ell=1}^{r^*} q_\ell^*, \mathbf{i} - \sum_{\ell=1}^{r^*-1} q_\ell^*, \dots, \mathbf{i} - q_1^*, \mathbf{i} + p_{r^*}^*, \mathbf{i} + p_{r^*-1}^*, \dots, \mathbf{i} + p_1^* \right\} \setminus \{0, n+1\}$$

and $\mathbf{a} = r^* - 1$ is the number of interior corners of π^* . Using formulas for (p_j^*, q_j^*) that precede the proposition, we recover the expression for \mathbf{J} given in the statement. \square

Remark. The Schubert varieties in any classical CHSS admit a partition description; and there exist analogous recipes relating the partitions to the (\mathbf{a}, \mathbf{J}) data.

4. THE SCHUBERT SYSTEM \mathcal{B}_w

4.1. Definition. Corollary 8.2 of [14] asserts that $\bigwedge^k \mathfrak{g}_{-1}$ decomposes as a direct sum of irreducible \mathfrak{g}_0 -modules

$$(4.1) \quad \bigwedge^k \mathfrak{g}_{-1} = \bigoplus_{w \in W^{\mathfrak{p}}(k)} \mathbf{I}_w.$$

Above, $W^{\mathfrak{p}}(k)$ is the set of elements of the Hasse diagram $W^{\mathfrak{p}}$ of length k and \mathbf{I}_w is the irreducible \mathfrak{g}_0 -module of highest weight $w(\rho) - \rho$. Let $X \subset \mathbb{P}V_{\omega_{\mathbf{i}}}$ be an irreducible compact Hermitian symmetric space in its minimal homogeneous embedding. By (2.2) and (3.1), $T_o X \simeq \mathfrak{g}_{-1}$ as \mathfrak{g}_0 -modules. So (4.1) determines a \mathfrak{g}_0 -module decomposition of $\bigwedge^k T_o X$.

The $|w|$ -plane $\mathbf{n}_w \in \text{Gr}(|w|, \mathfrak{g}_{-1})$ is a highest weight line of \mathbf{I}_w . Define B_w to be the G_0 -orbit of $\mathbf{n}_w \in \text{Gr}(|w|, \mathfrak{g}_{-1})$. Under the \mathfrak{p} -module identification $T_o X \simeq \mathfrak{g}/\mathfrak{p}$ the abelian \mathfrak{g}_+ acts trivially, and B_w is P -stable. The orbit $B_w \subset \text{Gr}(|w|, \mathfrak{g}_{-1})$ is precisely the set of tangent spaces $T_o(p \cdot X_w)$, $p \in P$.

Given $z = gP \in X$, the *Schubert system* $\mathcal{B}_w \subset \text{Gr}(|w|, TX)$ is defined by $\mathcal{B}_{w,z} := g_* B_w \in \text{Gr}(|w|, T_z X)$. The fact that B_w is stable under P implies that $\mathcal{B}_{w,z}$ is well-defined. A $|w|$ -dimensional complex submanifold $M \subset X$ is an *integral manifold of \mathcal{B}_w* if $TM \subset \mathcal{B}_w$. A subvariety $Y \subset X$ is an *integral variety of \mathcal{B}_w* if the smooth locus $Y^0 \subset Y$ is an integral manifold. The Schubert system is *rigid* if for every integral manifold M , there exists $g \in G$ such that $g \cdot M \subset X_w$. If every integral variety Y of \mathcal{B}_w is of the form $g \cdot X_w$, then we say X_w is *Schubert rigid*.

Remark. If Y is an (irreducible) integral variety of \mathcal{B}_w , then its smooth locus Y^0 is a (connected) integral manifold of \mathcal{B}_w whose closure is Y . If \mathcal{B}_w is rigid, then there is some $g \in G$ such that $Y^0 \subset g \cdot X_w$ is open. Since $g \cdot X_w$ is irreducible, then the closure of Y^0 is $g \cdot X_w$, and hence $Y = g \cdot X_w$. Thus, if \mathcal{B}_w is rigid, then X_w is Schubert rigid.

The first step in our analysis of \mathcal{B}_w is to lift the problem up to a frame bundle \mathcal{G} over $X \subset \mathbb{P}V_{\omega_1}$.

4.2. A frame bundle. Set $V = V_{\omega_1}$ and $\dim V = N + 1$. Given $v \in V \setminus \{0\}$, let $[v] \in \mathbb{P}V$ denote the corresponding point in projective space. Given any subset $Y \subset \mathbb{P}V$, let $\widehat{Y} = \{v \in V \setminus \{0\} \mid [v] \in Y\}$ denote the cone over Y . Suppose that $y \in Y$ is a smooth point and $v \in \widehat{y}$. Then the tangent space $T_v \widehat{Y}$ (an intrinsic object) may be naturally identified with a linear subspace $\widehat{T}_y Y \subset V$ (an extrinsic object). If $X \subset \mathbb{P}V$ is the minimal homogeneous embedding of G/P , then $o \in G/P$ is naturally identified with $[v_0] \in \mathbb{P}V$, where $v_0 \in V \setminus \{0\}$ is a highest weight vector. Because v_0 is a highest weight vector we have

$$(4.2) \quad \widehat{T}_o X = \mathfrak{g} \cdot v_0 = \mathbb{C}v_0 \oplus \mathfrak{g}_{-1} \cdot v_0.$$

Fix a basis $\mathbf{v} = (v_0, v_1, \dots, v_N)$ for V , so that

$$\text{span}\{v_1, \dots, v_{|w|}\} = \mathfrak{n}_w \cdot v_0.$$

Define \mathcal{G} to be the set of frames (bases)

$$\mathcal{G} := \{g \cdot \mathbf{v} \mid g \in G\}.$$

Note that the bundle \mathcal{G} is naturally identified with the Lie group G . Elements of \mathcal{G} are sometimes expressed as $v = (v_0, \dots, v_N)$ where $v_i = g \cdot v_i$. The set of frames \mathcal{G} is a right P -bundle over X under the projection

$$\pi : \mathcal{G} \rightarrow X$$

mapping $v = (v_0, \dots, v_N)$ to $[v_0]$. (Alternatively, under the identification $\mathcal{G} \simeq G$, π maps $g \in G$ to $[g \cdot v_0] \in \mathbb{P}V$.)

4.3. The Maurer-Cartan form. Let ϑ denote the \mathfrak{g} -valued, left-invariant *Maurer-Cartan 1-form* on \mathcal{G} . At a point $v = g \cdot \mathbf{v} \in \mathcal{G}$, $\vartheta_v = \vartheta_g$ is defined by $dv_j = \vartheta_j^i v_i = \vartheta \cdot v_j$. Regard each v_i as a map $G \rightarrow V$ sending $g \mapsto g \cdot v_i$. Then (4.2) yields

$$(4.3) \quad dv_0 \equiv \text{Ad}_g(\vartheta_{\mathfrak{g}_{-1}}) \cdot v_0 \pmod{\text{span}\{v_0\}}.$$

In particular, the semi-basic forms for the projection $\mathcal{G} \rightarrow X$ at $v = g \cdot \mathbf{v}$ are spanned by the $\vartheta_{\mathfrak{g}_{-1}}$.

Given \mathfrak{g} -valued 1-forms φ and χ , let $[\varphi, \chi]$ denote the \mathfrak{g} -valued 2-form given by

$$[\varphi, \chi](u, v) := [\varphi(u), \chi(v)] - [\varphi(v), \chi(u)].$$

Note that $[\varphi, \chi] = [\chi, \varphi]$; we make frequent use of this in computations without mention. The derivative of the Maurer-Cartan form is given by the *Maurer-Cartan equation*

$$(4.4) \quad d\vartheta = -\frac{1}{2}[\vartheta, \vartheta].$$

Given any \mathfrak{g} -valued form φ on \mathcal{G} and a direct sum decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, let $\varphi_{\mathfrak{a}}$ denote the component of φ taking value in \mathfrak{a} at $v \in \mathcal{G}$. The following lemma is classical.

Lemma 4.5. *Let $\mathfrak{a} \subset \mathfrak{g}$ be a subalgebra and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ vector space direct sum decomposition. Set $\mathcal{A} = \exp(\mathfrak{a}) \cdot v \subset \mathcal{G}$. The system $\vartheta_{\mathfrak{a}^\perp} = 0$ is Frobenius. The maximal integral manifolds are $g \cdot \mathcal{A} \subset \mathcal{G}$, where $g \in G$ is fixed.*

Proof. To see that $\vartheta_{\mathfrak{a}^\perp} = 0$ is Frobenius note that the Maurer-Cartan equation implies $d\vartheta_{\mathfrak{a}^\perp} = -\frac{1}{2} [\vartheta, \vartheta]_{\mathfrak{a}^\perp}$. Using $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ and $\vartheta = \vartheta_{\mathfrak{a}} + \vartheta_{\mathfrak{a}^\perp}$ we have $d\vartheta_{\mathfrak{a}^\perp} \equiv 0$ modulo $\vartheta_{\mathfrak{a}^\perp}$; the system is Frobenius. The remainder of the lemma follows from Cartan's Theorem [11, Theorem 1.6.10] on Lie algebra valued 1-forms satisfying the Maurer-Cartan equation. \square

To simplify notation set $\vartheta_{j,k} := \vartheta_{\mathfrak{g}_{j,k}}$.

4.4. Lifting the Schubert system to the frame bundle. Let $M \subset X$ be a (connected) complex submanifold of dimension $|w|$. (We are interested in the case that M is the set of smooth points Y^0 of a variety Y .) By definition M is an integral manifold of \mathcal{B}_w if and only if $TM \subset \mathcal{B}_w$. Equivalently, every point in M admits an open neighborhood $U \subset M$ with local section $\sigma : U \rightarrow \mathcal{G}$ such that $\{\sigma_0(u), \dots, \sigma_{|w|}(u)\}$ spans $\hat{T}_u M$ for all $u \in U$.

Define the *adapted frame bundle* \mathcal{F} over an integral manifold M of \mathcal{B}_w to be

$$(4.6) \quad \mathcal{F} := \left\{ v \in \mathcal{G} : [v_0] \in M, \hat{T}_{[v_0]} M = \text{span}\{v_0, \dots, v_{|w|}\} \right\}.$$

Note that \mathcal{F} is a right P_w -bundle over M , where $P_w \subset P$ is the parabolic subgroup preserving the flag $\mathbb{C}v_0 \subset \mathbb{C}v_0 \oplus \mathfrak{n}_w \cdot v_0 \subset V$.

Since $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the stabilizer of $\mathbb{C}v_0$, the Lie algebra of P_w is the subalgebra $\mathfrak{p}_w \subset \mathfrak{p}$ stabilizing $\mathbb{C}v_0 \oplus \mathfrak{n}_w \cdot v_0$. Given Proposition 3.9, it is straight-forward to see that $\mathbb{C}v_0 \oplus \mathfrak{n}_w \cdot v_0$ is stabilized by $\mathfrak{g}_{0, \geq 0}$ in \mathfrak{g}_0 and by all of \mathfrak{g}_1 . Thus

$$(4.7) \quad \mathfrak{p}_w = \mathfrak{g}_{0, \geq 0} \oplus \mathfrak{g}_1.$$

Observe that (4.3) and (4.6) imply $\vartheta_{\mathfrak{n}_w^\perp} = 0$ when restricted to \mathcal{F} , cf. (3.18). In particular, $\vartheta_{\mathfrak{n}_w}$ spans the semi-basic 1-forms on \mathcal{F} . Since $\vartheta_{\mathfrak{p}_w}$ spans the vertical 1-forms, it follows that $\vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w}$ is a coframing of \mathcal{F} .

Let $\{w_1, \dots, w_s\}$ be a basis of $\mathfrak{n}_w \oplus \mathfrak{p}_w$. Then $\vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w} = \theta^a w_a$ uniquely defines 1-forms θ^a , $a = 1, \dots, s$. Set $\wedge \vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w} := \theta^1 \wedge \dots \wedge \theta^s$. While the form $\wedge \vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w}$ depends on our choice of basis, its vanishing (or nonvanishing) on any s -dimensional tangent subspace does not. The statement that $\vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w}$ is a coframing on \mathcal{F} is equivalent to the statement that $\wedge \vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w}$ is nowhere vanishing on \mathcal{F} .

Definition. The *Schubert system* on \mathcal{G} is the linear Pfaffian system

$$(4.8) \quad \vartheta_{\mathfrak{n}_w^\perp} = 0 \quad \text{with} \quad \wedge \vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w} \neq 0.$$

An *integral manifold* of (4.8) is a submanifold $\mathcal{M} \subset \mathcal{G}$ such that $\dim \mathcal{M} = \dim \mathfrak{n}_w \oplus \mathfrak{p}_w$, $\vartheta_{\mathfrak{n}_w^\perp}$ vanishes when restricted to \mathcal{M} , and $\wedge \vartheta_{\mathfrak{n}_w \oplus \mathfrak{p}_w}$ is nowhere zero on \mathcal{M} .

Lemma 4.9. *The integral manifolds of (4.8) are precisely the adapted bundles \mathcal{F} over integral manifolds M of the Schubert system \mathcal{B}_w .*

Proof. First assume that \mathcal{F} is adapted frame bundle over an integral manifold M of \mathcal{B}_w . That \mathcal{F} is an integral manifold of (4.8) follows directly from the discussion following (4.7).

Now assume that \mathcal{M} is an integral manifold of (4.8). Set $M = \pi(\mathcal{M})$. We will show that (1) M is a submanifold of X , (2) M is an integral manifold of \mathcal{B}_w , and (3) \mathcal{M} is the

adapted bundle (4.6) over M . The independence condition and the vanishing of $\vartheta_{\mathfrak{n}_w^\perp}$ imply that $\pi : \mathcal{M} \rightarrow X$ is of constant rank $|w|$: so every $e \in \mathcal{M}$ admits a neighborhood $U \subset \mathcal{M}$ such that $\pi(U)$ is a submanifold of X . To see that M , the union of the various $\pi(U)$, is itself a submanifold of X we need the following.

Claim 4.10. The form $\vartheta_{0,-}$ vanishes when restricted to fibres of \mathcal{M} .

The claim is proven below. Notice that the fibre-wise vanishing of $\vartheta_{0,-}$ implies that the linear span of $\{v_0, v_1, \dots, v_{|w|}\}$ is constant on the fibre $\mathcal{M}_{[v_0]}$ over $[v_0] \in M$. Thus $\widehat{T}_{[v_0]}M$ is well-defined and M is a submanifold of X .

That M is an integral manifold of \mathcal{B}_w is a consequence of $\vartheta_{\mathfrak{n}_w^\perp} = 0$. Finally, it follows from (4.8) and the definition (4.6) that \mathcal{M} is the adapted frame bundle \mathcal{F} over M . \square

Proof of Claim 4.10. The fibre-wise vanishing of $\vartheta_{0,-}$ is equivalent to $\vartheta_{0,-} \equiv 0$ modulo $\vartheta_{\mathfrak{n}_w}$ on \mathcal{F} . Differentiating $\vartheta_{\mathfrak{n}_w^\perp} = 0$ with the Maurer-Cartan equation (4.4) yields

$$(4.11) \quad 0 = -d\vartheta_{\mathfrak{n}_w^\perp} = [\vartheta_{0,-}, \vartheta_{\mathfrak{n}_w}]_{\mathfrak{n}_w^\perp}.$$

Suppose that $\xi \in \mathfrak{g}_{0,-}$ and $[\xi, \mathfrak{n}_w]_{\mathfrak{n}_w^\perp} = \{0\}$. Then ξ preserves \mathfrak{n}_w and (3.8) implies $\xi = 0$. We may apply Cartan's Lemma [4, 11] to conclude that $\vartheta_{0,-} \equiv 0$ modulo $\vartheta_{\mathfrak{n}_w}$. \square

4.5. Distinguishing Schubert varieties. Now consider the case that $Y = X_w$ is a Schubert variety, and review the definitions (3.10) and (3.18). Let $N_w \subset G_w$ be the connected Lie subgroups of G with Lie algebras $\mathfrak{n}_w \subset \mathfrak{g}_w$. The orbit $N_w \cdot o \subset X$ is a dense subset of the smooth points X_w^0 of X_w , and $\mathcal{G}_w := G_w \cdot \mathfrak{v} \subset \mathcal{G}$ is a sub-bundle of the adapted frames (4.6) over $N_w \cdot o$.

When restricted to \mathcal{G}_w , the Maurer-Cartan form takes values in \mathfrak{g}_w . In particular, $\mathcal{G}_w \subset \mathcal{G}$ is a maximal integral submanifold of the system $\vartheta_{\mathfrak{g}_w^\perp} = 0$. Conversely, Lemma 4.5 implies that every integral manifold of

$$(4.12) \quad 0 = \vartheta_{\mathfrak{g}_w^\perp} \quad \text{with} \quad \bigwedge \vartheta_{\mathfrak{g}_w} \neq 0$$

is an open submanifold of $\mathcal{G}_w := G_w \cdot \mathfrak{v}$ (modulo the action of G on \mathcal{G}). This establishes the following.

Proposition 4.13. *The Schubert system \mathcal{B}_w is rigid if and only if every integral manifold \mathcal{F} of (4.8) admits a sub-bundle on which $\vartheta_{\mathfrak{g}_w^\perp}$ vanishes.*

4.6. Torsion. In proving Claim 4.10, we observed that $\vartheta_{0,-} \equiv 0$ modulo $\vartheta_{\mathfrak{n}_w}$ on any integral manifold \mathcal{F} of (4.8). Equivalently, there exists a function $\lambda : \mathcal{F} \rightarrow \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^*$ such that

$$(4.14) \quad \vartheta_{0,-} = \lambda(\vartheta_{\mathfrak{n}_w})$$

on \mathcal{F} . The λ are not arbitrary – they are constrained by torsion. To identify the torsion constraints substitute (4.14) into (4.11). We find

$$(4.15) \quad 0 = \left([u, \lambda(v)] - [v, \lambda(u)] \right)_{\mathfrak{n}_w^\perp}, \quad \forall u, v \in \mathfrak{n}_w.$$

Equivalently, λ must lie in the kernel of the map

$$(4.16) \quad \delta^1 : \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^* \rightarrow \mathfrak{n}_w^\perp \otimes \bigwedge^2 \mathfrak{n}_w^*$$

defined by

$$(\delta^1 \lambda) u \wedge v := \left([u, \lambda(v)] - [v, \lambda(u)] \right)_{\mathfrak{n}_w^\perp}, \quad u, v \in \mathfrak{n}_w.$$

These are the torsion constraints on λ .

4.7. Fibre motions in \mathcal{F} . Let \mathcal{F} be a maximal integral manifold of the Schubert system (4.8). In this section we investigate the variation of λ along the fibres of \mathcal{F} . It is often possible to normalize (components of) λ to zero via fibre motions within \mathcal{F} . Equivalently, \mathcal{F} will admit a sub-bundle on which λ (equivalently, $\vartheta_{0,-}$) vanishes.

These normalizations are obtained as follows. Recall that \mathcal{F} is a right P_w -bundle. Any smooth function $p : \mathcal{F} \rightarrow P_w$ naturally induces a bundle map $\bar{p} : \mathcal{F} \rightarrow \mathcal{F}$ mapping $e \in \mathcal{F}$ to $e \cdot p(e)$. There are two types of fibre motions to consider: those with values in $G_{0,\geq 0} = \exp(\mathfrak{g}_{0,\geq 0}) \subset P_w$, and those with values in $G_1 = \exp(\mathfrak{g}_1) \subset P_w$. The $G_{0,\geq 0}$ -valued fibre motions are essentially changes of coordinates and so can not be used to normalize λ to zero; see the proof of Lemma 4.21. So we assume that p takes value in G_1 .

Let ϱ be the left-invariant, \mathfrak{g}_1 -valued Maurer-Cartan form on G_1 . Then

$$\bar{p}^* \vartheta_{\bar{p}(e)} = p^* \varrho_{p(e)} + \text{Ad}_{p(e)^{-1}} \vartheta_e.$$

Pick $t : \mathcal{F} \rightarrow \mathfrak{g}_1$ with $\exp(-t) = p^{-1}$. Dropping the base point e from our notation, and recalling that $\text{Ad} \circ \exp = \exp \circ \text{ad}$, we have

$$\begin{aligned} (\bar{p}^* \vartheta)_{\mathfrak{n}_w} &= (\text{Ad}_{p^{-1}} \vartheta)_{\mathfrak{n}_w} = \vartheta_{\mathfrak{n}_w} \\ (\bar{p}^* \vartheta)_{0,-} &= (\text{Ad}_{p^{-1}} \vartheta)_{0,-} = \vartheta_{0,-} - (\text{ad}_t \cdot \vartheta_{\mathfrak{n}_w})_{0,-}. \end{aligned}$$

(The second line above, along with $(\bar{p}^* \vartheta)_{\mathfrak{n}_w^\perp} = 0$, confirms that \bar{p}^* preserves the EDS.) Since $t \in \mathfrak{g}_1$ and $\mathfrak{n}_w \subset \mathfrak{g}_{-1}$ we may regard ad_t as an element of $\mathfrak{g}_0 \otimes \mathfrak{n}_w^*$. Let $\text{ad}_{t,-}$ be the projection of ad_t to $\mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^*$. Then the second equation above reads $(\bar{p}^* \vartheta)_{0,-} = \vartheta_{0,-} - \text{ad}_{t,-}(\vartheta_{\mathfrak{n}_w})$. On the other hand, $(\bar{p}^* \vartheta)_{0,-} = \bar{p}^* \lambda(\vartheta_{\mathfrak{n}_w}) = (\lambda \circ \bar{p}) \vartheta_{\mathfrak{n}_w}$. Thus,

$$(4.17) \quad \bar{p}^* \lambda = \lambda \circ \bar{p} = \lambda - \text{ad}_{t,-}.$$

In particular, $\lambda - \bar{p}^* \lambda$ lies in the image of the $\mathfrak{g}_{0,0}$ -module map

$$(4.18) \quad \delta^0 : \mathfrak{g}_1 \rightarrow \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^*$$

defined by

$$(4.19) \quad (\delta^0 t)u := [t, u]_{0,-},$$

where $t \in \mathfrak{g}_1$ and $u \in \mathfrak{n}_w$.

Lemma 4.20. *The kernel of δ^0 is $\mathfrak{g}_{1,\geq a} \subset \mathfrak{g}_w$. Thus, $\delta^0 : \mathfrak{g}_{1,<a} \rightarrow \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^*$ is injective.*

Proof. It is clear from Proposition 3.9 that $\mathfrak{g}_{1,\geq a} \subset \ker \delta^0$. So suppose that $t \in \mathfrak{g}_{1,<a}$ lies in the kernel of δ^0 . That is, $[t, u]_{0,-} = 0$ for all $u \in \mathfrak{n}_w$. It suffices to show that $t = 0$. Express t as a linear combination $t = \sum_{\alpha \in \Delta(\mathfrak{g}_{1,<a})} t^\alpha E_\alpha$ of root vectors E_α in $\mathfrak{g}_{1,<a}$ with coefficients $t^\alpha \in \mathbb{C}$.

Observe that $\delta^0(t)(\mathfrak{g}_{-1,-1}) = [t_0, \mathfrak{g}_{-1,-1}] \in \mathfrak{g}_{0,-1}$, where t_a is the component of t taking values in $\mathfrak{g}_{1,a}$. So $t \in \ker \delta^0$ forces $[E_\alpha, \mathfrak{g}_{-1,-1}] = \{0\} \subset \mathfrak{g}_{0,-1}$ for every $\alpha \in \Delta(\mathfrak{g}_{1,0})$ with $t^\alpha \neq 0$. This in turn implies that $E_\alpha \notin [\mathfrak{g}_{0,-1}, \mathfrak{g}_{1,1}]$. This is a contradiction as every root

space \mathfrak{g}_α is obtained from the highest root space $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{g}_{1,\mathfrak{m}}$ by successive brackets with $\mathfrak{g}_{-\alpha_j} \subset \mathfrak{g}_{0,-1}$, with α_j simple. It follows that $t_0 = 0$.

The claim that $t = 0$ now follows by induction: assume that $t_a = 0$ for all $a < a_0$. Then, as above, $\{0\} = \delta^0(t)(\mathfrak{g}_{-1,-1-a_0}) = [t_{a_0}, \mathfrak{g}_{-1,-1-a_0}] \subset \mathfrak{g}_{0,-1}$ forces $t_{a_0} = 0$. \square

Next we claim that $\text{im } \delta^0 \subset \ker \delta^1$. To see this compute

$$\begin{aligned} \delta^1 \circ \delta^0(t)(u \wedge v) &= [u, \delta^0(t)v]_{\mathfrak{n}_w^\perp} - [v, \delta^0(t)u]_{\mathfrak{n}_w^\perp} \\ &= [u, [t, v]_{0,-}]_{\mathfrak{n}_w^\perp} - [v, [t, u]_{0,-}]_{\mathfrak{n}_w^\perp} \\ &= [u, [t, v]]_{\mathfrak{n}_w^\perp} - [v, [t, u]]_{\mathfrak{n}_w^\perp} \stackrel{(\star)}{=} [t, [u, v]]_{\mathfrak{n}_w^\perp} \stackrel{(\dagger)}{=} 0. \end{aligned}$$

The equality (\star) is the Jacobi identity; and the equality (\dagger) follows from (3.2).

Lemma 4.21. *Let $\ker \delta^1 = \text{im } \delta^0 \oplus (\text{im } \delta^0)^\perp$ be a $\mathfrak{g}_{0,0}$ -module decomposition. Then any integral manifold \mathcal{F} of (4.8) admits a normalization, via the map \bar{p} , to a sub-bundle $\mathcal{F}^0 := \bar{p}(\mathcal{F})$ on which λ takes values in $(\text{im } \delta^0)^\perp$. Additionally, the $\vartheta_{\mathfrak{g}_w}$ compose a coframing of \mathcal{F}^0 and there exists $\mu : \mathcal{F}^0 \rightarrow \mathfrak{g}_{1,<a} \otimes \mathfrak{n}_w^*$ such that $\vartheta_{1,<a} = \mu(\vartheta_{\mathfrak{n}_w})$.*

Proof. The first statement follows from the observations above; it remains to establish the second half of the lemma. Let $\mathcal{F}^0 \subset \mathcal{F}$ denote the sub-bundle on which the normalization holds. From Lemma 4.20 it follows that $\vartheta_{1,<a}$ vanishes when pulled back to the fibres of \mathcal{F}^0 . Equivalently, $\vartheta_{1,<a}$ is semi-basic: there exists $\mu : \mathcal{F}^0 \rightarrow \mathfrak{g}_{1,<a} \otimes \mathfrak{n}_w^*$ such that $\vartheta_{1,<a} = \mu(\vartheta_{\mathfrak{n}_w})$.

To see that the $\vartheta_{\mathfrak{g}_w}$ are linearly independent on \mathcal{F}^0 and thus a coframing (cf. Lemma 4.9), it suffices to observe that the fibre motions associated to $\mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,\geq a} \subset \mathfrak{g}_w$ preserve $\text{im } \delta^0$. This is immediate for fibre motions in the directions of $\mathfrak{g}_{1,\geq a}$ by Lemma 4.20. In the case that $t : \mathcal{F} \rightarrow \mathfrak{g}_{0,\geq 0}$ a computation similar to that above for (4.17) shows that

$$(\bar{p}^* \lambda)(u - [t, u]) = (\lambda \circ \bar{p})(u - [t, u]) = \lambda(u) - [t, \lambda(u)]_{0,-}$$

for all $u \in \mathfrak{n}_w$. Thus, $\text{im } \delta^0$ is preserved under fibre motions in the directions of $\mathfrak{g}_{0,\geq 0}$. \square

4.8. When λ can be normalized to zero. Assume \mathcal{F} admits a sub-bundle \mathcal{F}^0 on which λ vanishes. Equivalently, $\vartheta_{0,-} = 0$ on \mathcal{F}^0 . Thus,

$$(4.22) \quad 0 = -d\vartheta_{0,-} = \frac{1}{2} [\vartheta, \vartheta]_{0,-} = [\vartheta_{\mathfrak{n}_w}, \vartheta_{1,<a}]_{0,-}.$$

By Lemma 4.21, $\vartheta_{1,<a} = \mu(\vartheta_{\mathfrak{n}_w^\perp})$. Substituting this into (4.22) yields torsion, forcing μ to lie in the kernel of the map

$$(4.23) \quad \varepsilon^1 : \mathfrak{g}_{1,<a} \otimes \mathfrak{n}_w^* \rightarrow \mathfrak{g}_{0,-} \otimes \wedge^2 \mathfrak{n}_w^*,$$

defined by

$$\varepsilon^1(\mu)u \wedge v := [u, \mu(v)]_{0,-} - [v, \mu(u)]_{0,-}.$$

Definition (3.18), Proposition 4.13, Lemma 4.21 and the discussion above yield the following key observation.

4.9. Lie algebra cohomology. Let \mathfrak{a} be a Lie algebra and Γ a \mathfrak{a} -module. Then the Lie algebra cohomology differential $\partial = \partial^k : \Gamma \otimes \bigwedge^k \mathfrak{a}^* \rightarrow \Gamma \otimes \bigwedge^{k+1} \mathfrak{a}^*$ is defined as follows. Given $\phi \in \Gamma \otimes \bigwedge^k \mathfrak{a}^*$ and $u_0, \dots, u_k \in \mathfrak{a}$,

$$(4.24) \quad \begin{aligned} (\partial\phi)(u_0, \dots, u_k) &:= \sum_i (-1)^i u_i \cdot \phi(u_0, \dots, \widehat{u}_i, \dots, u_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} \phi([u_i, u_j], u_0, \dots, \widehat{u}_i, \dots, \widehat{u}_j, \dots, u_k). \end{aligned}$$

The differential satisfies $\partial \circ \partial = 0$ and so defines cohomology groups

$$H^k(\mathfrak{a}, \Gamma) := \frac{\ker\{\partial : \Gamma \otimes \bigwedge^k \mathfrak{a}^* \rightarrow \Gamma \otimes \bigwedge^{k+1} \mathfrak{a}^*\}}{\text{im}\{\partial : \Gamma \otimes \bigwedge^{k-1} \mathfrak{a}^* \rightarrow \Gamma \otimes \bigwedge^k \mathfrak{a}^*\}}.$$

From (3.2) and $\mathfrak{n}_w \subset \mathfrak{g}_{-1}$ we see that \mathfrak{n}_w is an abelian Lie algebra. Given $A \in \mathfrak{g}$, let $A_{\mathfrak{g}_w^\perp}$ denote the image of A under the natural projection $\mathfrak{g} = \mathfrak{g}_w \oplus \mathfrak{g}_w^\perp \rightarrow \mathfrak{g}_w^\perp$. Define an action of \mathfrak{n}_w on \mathfrak{g}_w^\perp by

$$u \cdot z := [u, z]_{\mathfrak{g}_w^\perp}, \quad u \in \mathfrak{n}_w, \quad z \in \mathfrak{g}_w^\perp.$$

Making use of (3.10), (3.18) and the Jacobi identity it is straight-forward to confirm that this action respects the Lie algebra structure of \mathfrak{n}_w ; that is, \mathfrak{g}_w^\perp is a \mathfrak{n}_w -module.

In the case that $\mathfrak{a} = \mathfrak{n}_w$ and $\Gamma = \mathfrak{g}_w^\perp$, the differential ∂ is a $\mathfrak{g}_{0,0}$ -module map. Thus $H^k(\mathfrak{n}_w, \mathfrak{g}_w^\perp)$ is a $\mathfrak{g}_{0,0}$ -module, and therefore admits a Z_1 -graded decomposition. For $k = 1$,

$$(4.25) \quad H^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp) = H_0^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp) \oplus H_1^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp) \oplus H_2^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp).$$

Here $H_m^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp)$ is the Z_1 -eigenspace associated to the eigenvalue m .

Keeping in mind that \mathfrak{n}_w is abelian, we see that

$$(4.26) \quad H_1^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp) = \frac{\ker \delta^1}{\text{im } \delta^0} \quad \text{and} \quad H_2^1(\mathfrak{n}_w, \mathfrak{g}_w^\perp) = \ker \varepsilon^1.$$

5. COHOMOLOGICAL COMPUTATIONS

5.1. The adjoint operators. In this section we construct an adjoint ∂^* to the Lie algebra differential ∂ of Section 4.9 for $\mathfrak{a} = \mathfrak{n}_w$ and $\Gamma = \mathfrak{g}_w^\perp$.

Define a linear map $\mathbf{d}^* : \mathfrak{g} \otimes \bigwedge^{k+1} \mathfrak{g} \rightarrow \mathfrak{g} \otimes \bigwedge^k \mathfrak{g}$ on decomposable elements $v \otimes (z_0 \wedge \dots \wedge z_k)$ by

$$\begin{aligned} \mathbf{d}^*(v \otimes (z_0 \wedge \dots \wedge z_k)) &:= \sum_i (-1)^i [z_i, v] \otimes (z_0 \wedge \dots \wedge \widehat{z}_i \wedge \dots \wedge z_k) \\ &\quad + \sum_{i < j} (-1)^{i+j+1} v \otimes ([z_i, z_j] \wedge z_0 \wedge \dots \wedge \widehat{z}_i \wedge \dots \wedge \widehat{z}_j \wedge \dots \wedge z_k). \end{aligned}$$

The Killing form (\cdot, \cdot) on \mathfrak{g} provides a canonical identification

$$(5.1) \quad \mathfrak{n}_w^* \simeq \mathfrak{n}_w^+ := \mathfrak{g}_{1,0} \oplus \dots \oplus \mathfrak{g}_{1,\mathfrak{a}} \quad \text{by} \quad u \in \mathfrak{n}_w^+ \mapsto (u, \cdot) \in \mathfrak{n}_w^*$$

as $\mathfrak{g}_{0,0}$ -modules. In particular,

$$(5.2) \quad \mathfrak{g}_w^\perp \otimes \bigwedge^k \mathfrak{n}_w^* \simeq \mathfrak{g}_w^\perp \otimes \bigwedge^k \mathfrak{n}_w^+ \subset \mathfrak{g} \otimes \bigwedge^k \mathfrak{g}.$$

Using this identification, we define $\partial^* : \mathfrak{g}_w^\perp \otimes \bigwedge^{k+1} \mathfrak{n}_w^* \rightarrow \mathfrak{g}_w^\perp \otimes \bigwedge^k \mathfrak{n}_w^*$ to be the restriction

$$(5.3) \quad \partial^* = \mathbf{d}^*|_{\mathfrak{g}_w^\perp \otimes \bigwedge^{k+1} \mathfrak{n}_w^+}.$$

A priori, the image of ∂^* may not lie in $\mathfrak{g}_w^\perp \otimes \bigwedge^k \mathfrak{n}_w^+$. However, (3.18) and (5.1) imply $[\mathfrak{g}_w^\perp, \mathfrak{n}_w^+] \subset \mathfrak{g}_w^\perp$. Finally, note that ∂^* is a $\mathfrak{g}_{0,0}$ -module map.

Proposition 5.4. *There exists a natural positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^*$ with respect to which the operators ∂ and ∂^* are adjoint.*

Proof. The inner product is defined as follows. Let $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ denote the Killing form on \mathfrak{g} . Let $\mathfrak{k} \subset \mathfrak{g}$ be a compact form of \mathfrak{g} . Set $i = \sqrt{-1}$. Define a conjugate-linear Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ by $\theta|_{\mathfrak{k}} = \text{Id}|_{\mathfrak{k}}$ and $\theta|_{i\mathfrak{k}} = -\text{Id}|_{i\mathfrak{k}}$. It is straight-forward to check that $\theta([u, v]) = [\theta u, \theta v]$, for all $u, v \in \mathfrak{g}$. It is well-known that \mathfrak{k} may be selected so that $\theta(Z_j) = -Z_j$ and $\theta(\mathfrak{g}_\beta) = \mathfrak{g}_{-\beta}$ for all roots β . (See [5, Section 2.3].) Then

$$(5.5) \quad \langle u, v \rangle := -(u, \theta v)$$

defines a (positive definite) Hermitian inner product on \mathfrak{g} . Moreover,

$$(5.6) \quad \langle [z, u], v \rangle = -([z, u], \theta v) = (u, [z, \theta v]) = (u, \theta([\theta z, v])) = -\langle u, [\theta z, v] \rangle.$$

Thus $\text{ad}(z)^* = -\text{ad}(\theta z)$ is the adjoint of $\text{ad}(z) \in \mathfrak{gl}(\mathfrak{g})$ with respect to $\langle \cdot, \cdot \rangle$. (Alternatively, $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ is skew-Hermitian.) We also observe that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$, for all roots $\alpha \neq \beta$, and $\langle \mathfrak{h}, \mathfrak{g}_\beta \rangle = 0$ for all β . Hence,

$$(5.7) \quad \langle \mathfrak{g}_w^\perp, \mathfrak{g}_w \rangle = 0.$$

Abusing notation, we also let $\langle \cdot, \cdot \rangle$ denote the induced Hermitian inner product on $\mathfrak{g} \otimes \wedge^k \mathfrak{g}$, and its restriction to $\mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^*$ under the identification (5.2). Proposition 5.4 will now follow from the lemma below. \square

Lemma 5.8. *∂ and ∂^* are adjoint with respect to $\langle \cdot, \cdot \rangle$.*

Proof. Let $\Phi \in \mathfrak{g}_w^\perp \otimes \wedge^{k+1} \mathfrak{n}_w^*$. Choose $\gamma \in \mathfrak{g}_w^\perp$ and $u_0, \dots, u_k \in \mathfrak{n}_w$. Set $z_j = -\theta(u_j)$. By construction we have

$$(5.9) \quad \langle \Phi, \gamma \otimes (z_0 \wedge \dots \wedge z_k) \rangle_{\mathfrak{g}_w^\perp \otimes \wedge^{k+1} \mathfrak{n}_w^*} = \langle \Phi(u_0, \dots, u_k), \gamma \rangle_{\mathfrak{g}_w^\perp}.$$

Applying (5.9) with $\Phi = \partial\phi$ yields

$$\begin{aligned} \langle \partial\phi, \gamma \otimes (z_0 \wedge \dots \wedge z_k) \rangle &= \langle \partial\phi(u_0, \dots, u_k), \gamma \rangle \\ &\stackrel{(4.24)}{=} \left\langle \sum_{i=0}^k (-1)^i [u_i, \phi(u_0, \dots, \widehat{u}_i, \dots, u_k)]_{\mathfrak{g}_w^\perp}, \gamma \right\rangle \\ &\stackrel{(5.7)}{=} \left\langle \sum_{i=0}^k (-1)^i [u_i, \phi(u_0, \dots, \widehat{u}_i, \dots, u_k)], \gamma \right\rangle. \end{aligned}$$

On the other hand

$$\begin{aligned}
\langle \phi, \partial^*(\gamma \otimes (z_0 \wedge \cdots \wedge z_k)) \rangle &= \left\langle \phi, \sum_{i=0}^k (-1)^i [z_i, \gamma] \otimes (z_0 \wedge \cdots \wedge \widehat{z_i} \wedge \cdots \wedge z_k) \right\rangle \\
&\stackrel{(5.9)}{=} \sum_{i=0}^k (-1)^i \left\langle \phi(u_0, \dots, \widehat{u_i}, \dots, u_k), [z_i, \gamma] \right\rangle \\
&= \sum_{i=0}^k (-1)^{i+1} \left\langle \phi(u_0, \dots, \widehat{u_i}, \dots, u_k), [\theta(u_i), \gamma] \right\rangle \\
&\stackrel{(5.6)}{=} \sum_{i=0}^k (-1)^i \left\langle [u_i, \phi(u_0, \dots, \widehat{u_i}, \dots, u_k)], \gamma \right\rangle.
\end{aligned}$$

The lemma follows. \square

The *Laplacian* is the $\mathfrak{g}_{0,0}$ -module map defined by

$$\square := \partial \partial^* + \partial^* \partial : \mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^* \rightarrow \mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^*.$$

Let $\mathcal{H}^k := \ker \square \subset \mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^*$. By now standard arguments (see [14, Proposition 2.1] or [5, Corollary 3.3.1]) we have the following.

Proposition 5.10. *Each element in $H^k(\mathfrak{n}_w, \mathfrak{g}_w^\perp)$ admits a unique representative in \mathcal{H}^k . The corresponding identification $H^k(\mathfrak{n}_w, \mathfrak{g}_w^\perp) \simeq \mathcal{H}^k$ is a $\mathfrak{g}_{0,0}$ -module isomorphism.*

Let

$$(5.11) \quad \mathcal{H}^1 = \mathcal{H}_0^1 \oplus \mathcal{H}_1^1 \oplus \mathcal{H}_2^1 \subset (\mathfrak{g}_{-1, < -a} \oplus \mathfrak{g}_{0, -} \oplus \mathfrak{g}_{1, < a}) \otimes \mathfrak{n}_w^+.$$

denote the $Z_{\mathbf{i}}$ -graded decomposition of \mathcal{H}^1 .

Corollary 5.12. *Let \mathcal{F} be the adapted frame bundle (4.6) over an integral manifold of the Schubert system \mathcal{B}_w . The bundle \mathcal{F} admits a sub-bundle \mathcal{F}^0 on which $\lambda : \mathcal{F} \rightarrow \ker \delta^1$ restricts to take values in \mathcal{H}_1^1 . Additionally, the $\vartheta_{\mathfrak{g}_w}$ compose a coframing of \mathcal{F}^0 , and there exists $\mu : \mathcal{F}^0 \rightarrow \mathfrak{g}_{1, < a} \otimes \mathfrak{n}_w^+$ such that $\vartheta_{1, < a} = \mu(\vartheta_{\mathfrak{n}_w})$ on \mathcal{F}^0 .*

If λ vanishes on \mathcal{F}^0 , then μ takes value in \mathcal{H}_2^1 . If, for any integral manifold of \mathcal{B}_w , λ and μ vanish on \mathcal{F}^0 , then the Schubert system is rigid.

Proof. From (4.26) and Proposition 5.10 we see that

$$\ker \delta^1 = \text{im } \delta^0 \oplus \mathcal{H}_1^1 \quad \text{and} \quad \ker \varepsilon^1 = \mathcal{H}_2^1;$$

moreover, the former is a $\mathfrak{g}_{0,0}$ -module decomposition. Setting $(\text{im } \delta^0)^\perp := \mathcal{H}_1^1$, Lemma 4.21 yields a sub-bundle \mathcal{F}^0 on which λ takes values in \mathcal{H}_1^1 and $\vartheta_{1, < a} = \mu(\vartheta_{\mathfrak{n}_w})$.

If λ vanishes on \mathcal{F}^0 , then from Section 4.8 we see that μ takes values in \mathcal{H}_2^1 .

Finally, Proposition 4.13 asserts that the Schubert system is rigid if and only if every integral manifold \mathcal{F} of (4.8) admits a sub-bundle on which $\vartheta_{\mathfrak{g}_w^\perp}$ vanishes. Recall that $\mathfrak{g}_w^\perp = \mathfrak{n}_w^\perp \oplus \mathfrak{g}_{0, -} \oplus \mathfrak{g}_{1, < a}$. The vanishing of $\vartheta_{\mathfrak{n}_w^\perp}$ on $\mathcal{F}^0 \subset \mathcal{F}$ is automatic by Lemma 4.9. By (4.14), $\vartheta_{0, -}$ vanishes on \mathcal{F}^0 if and only if λ does. Similarly, the vanishing of $\vartheta_{1, < a}$ is equivalent to the vanishing of μ by Lemma 4.21. \square

(5.13) *From this point on, we restrict to the bundle \mathcal{F}^0 of Corollary 5.12.*

Remark. It is an immediate consequence of Corollary 5.12 that the Schubert system is rigid if $\mathcal{H}_+^1 = \{0\}$. We will see in Sections 5.4 & 5.5 that it suffices for subspaces $\mathcal{H}_{1,a-1}^1 \subset \mathcal{H}_1^1$ and $\mathcal{H}_{2,2a-1}^1 \subset \mathcal{H}_2^1$ to vanish. These subspaces are components of the (Z_i, Z_w) -bigraded decomposition

$$(5.14) \quad \mathcal{H}^1 = \oplus_{i,s} \mathcal{H}_{i,s}^1.$$

The integers $s = a - 1$ and $s = 2a - 1$ are respectively the maximal Z_w -eigenvalues on \mathcal{H}_1^1 and \mathcal{H}_2^1 .

5.2. Action of the Laplacian. In this section we derive general formulas, (5.19) and (5.21) respectively, that will be used to determine \mathcal{H}_1^1 and \mathcal{H}_2^1 .

Given $\xi \in \mathfrak{g}$, let $\epsilon_\xi : \mathfrak{g} \otimes \wedge^k \mathfrak{g} \rightarrow \mathfrak{g} \otimes \wedge^{k+1} \mathfrak{g}$ denote the natural map induced by exterior product with ξ . Let $\iota_\xi : \mathfrak{g} \otimes \wedge^k \mathfrak{g} \rightarrow \mathfrak{g} \otimes \wedge^{k-1} \mathfrak{g}$ denote the natural map induced by the interior product with ξ ; the interior product is computed with respect to the Killing form. Let $L'_\xi : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the adjoint action of ξ on \mathfrak{g} , and let $L''_\xi : \wedge \mathfrak{g} \rightarrow \wedge \mathfrak{g}$ denote the induced action on the exterior algebra. Then the induced action $\mathcal{L}_\xi : \mathfrak{g} \otimes \wedge \mathfrak{g} \rightarrow \mathfrak{g} \otimes \wedge \mathfrak{g}$ is given by $\mathcal{L}_\xi = \mathcal{L}'_\xi + \mathcal{L}''_\xi$ where $\mathcal{L}'_\xi = L'_\xi \otimes \mathbf{1}$ and $\mathcal{L}''_\xi = \mathbf{1} \otimes L''_\xi$.

The following lemma will be very useful.

Lemma 5.15 ([5, Lemma 3.3.2]). *Fix $\xi, \zeta \in \mathfrak{g}$, and let $\{\xi_\ell\}$ and $\{\zeta_\ell\}$ be Killing dual bases of \mathfrak{g} . Then \mathcal{L}'_ξ commutes with \mathcal{L}''_ζ , ϵ_ζ and ι_ζ ,*

- (a) $\mathbf{d}^* \circ \epsilon_\xi + \epsilon_\xi \circ \mathbf{d}^* = \mathcal{L}_\xi$,
- (b) $\mathcal{L}'_\xi \circ \mathbf{d}^* - \mathbf{d}^* \circ \mathcal{L}'_\xi = \sum_\ell \iota_{[\zeta_\ell, \xi]} \circ \mathcal{L}'_{\xi_\ell}$,
- (c) $\sum_\ell \epsilon_{\zeta_\ell} \circ \iota_{[\xi, \xi_\ell]} = -\mathcal{L}''_\xi$.

Remark. When consulting [5], note that their version of \mathbf{d}^* differs from ours by a sign.

Fix an orthogonal basis $\{H_1, \dots, H_n\}$ of \mathfrak{h} . Let $E_\alpha \in \mathfrak{g}_\alpha$ be root vectors scaled so that $(E_\alpha, E_{-\beta}) = \delta_{\alpha\beta}$. Then $\{\xi_\ell\} = \{E_\alpha\}_{\alpha \in \Delta} \cup \{H_j\}_{j=1}^n$ defines a basis of \mathfrak{g} . Let $\{\zeta_\ell\}$ denote the Killing dual basis.

Given $0 \leq a \in \mathbb{Z}$, set

$$\Delta(w, a) = \{\alpha \in \Delta(w) \mid \alpha(Z_w) = a\}.$$

From (3.2) and (4.24), we see that $\partial : \mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^+ \rightarrow \mathfrak{g}_w^\perp \otimes \wedge^{k+1} \mathfrak{n}_w^+$ is given by

$$(5.16) \quad \begin{aligned} \partial_{|\mathfrak{g}_{-1, -s} \otimes \wedge^k \mathfrak{n}_w^+} &= 0, \quad \text{for } s > a; \\ \partial_{|\mathfrak{g}_{0, -\ell} \otimes \wedge^k \mathfrak{n}_w^+} &= \sum_{\substack{\alpha \in \Delta(w, a) \\ a + \ell > a}} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}, \quad \text{for } \ell > 0; \\ \partial_{|\mathfrak{g}_{1, b} \otimes \wedge^k \mathfrak{n}_w^+} &= \sum_{\substack{\alpha \in \Delta(w, a) \\ b - a < 0}} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}, \quad \text{for } 0 \leq b < a. \end{aligned}$$

Bidegree $(1, k)$. The component of $\mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^* \simeq \mathfrak{g}_{0,-} \otimes \mathfrak{n}_w^+$ of (Z_i, Z_w) -bidegree $(1, k)$ is

$$(5.17a) \quad \Lambda^k := \bigoplus_{\substack{0 < \ell \\ k+\ell \leq a}} \Lambda_{-\ell}^k \quad \text{with} \quad \Lambda_{-\ell}^k := \mathfrak{g}_{0,-\ell} \otimes \mathfrak{g}_{1,k+\ell}, \quad k \leq a-1.$$

Note that $\mathcal{H}_{1,k}^1 = \Lambda^k \cap \mathcal{H}_1^1$, see (5.14). If $k + \ell < 0$, then $\mathfrak{g}_{1,k+\ell} = \{0\}$ so that

$$(5.17b) \quad \Lambda_{-\ell}^k = \{0\} \quad \text{when} \quad k + \ell < 0.$$

We have $\partial^*(\Lambda^k) \subset \mathfrak{g}_{1,k}$. Making use of (5.16), we compute

$$\begin{aligned} \square|_{\Lambda_{-\ell}^k} &= \sum_{\substack{\alpha \in \Delta(w,a) \\ a > k}} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* + \sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \\ &\stackrel{(*)}{=} \sum_{\substack{\alpha \in \Delta(w,a) \\ k < a \leq a-\ell}} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* + \sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \left(\partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* \right). \end{aligned}$$

Lemma 5.15(a,b) allows us to rewrite the second summand in $(*)$ as

$$\partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* = \mathcal{L}_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \sum_j \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j}.$$

Given $\alpha \in \Delta(w)$, (3.2) implies $\mathcal{L}_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} = \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}$ on $\mathfrak{g}_w^\perp \otimes \wedge^k \mathfrak{n}_w^+$. Also note that

$$(5.18) \quad \iota_\zeta \text{ will vanish on } \Lambda_{-\ell}^k \text{ unless } \zeta \in \mathfrak{g}_{-1, -k-\ell}.$$

Therefore, the second summand in $(*)$ may be further refined to

$$\partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* = \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ m=a-k-\ell}} \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j}.$$

Thus

$$\begin{aligned} \square|_{\Lambda_{-\ell}^k} &= \sum_{\substack{\alpha \in \Delta(w,a) \\ k < a \leq a-\ell}} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* + \sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \\ &\quad + \sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ m=a-k-\ell}} \epsilon_{E_\alpha} \circ \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j}. \end{aligned}$$

Swapping the order of summation in the third term above yields

$$\sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ m=a-k-\ell}} \epsilon_{E_\alpha} \circ \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j} = \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ a-k-2\ell < m \leq a-k-\ell}} \sum_{\substack{\alpha \in \Delta(w,a) \\ a=m+k+\ell}} \epsilon_{E_\alpha} \circ \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j}$$

Again, (5.18) allows us to replace the summation over $\{E_\alpha \mid \alpha \in \Delta(w, a), a = m + k + \ell\}$ with a summation over all ζ_p . Then Lemma 5.15(c) yields

$$\sum_{\substack{\alpha \in \Delta(w,a) \\ a+\ell > a}} \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ m=a-k-\ell}} \epsilon_{E_\alpha} \circ \iota_{[\zeta_j, E_{-\alpha}]} \circ \mathcal{L}'_{\xi_j} = - \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ a-k-2\ell < m \leq a-k-\ell}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}$$

The computations above yield the following expression for the Laplacian on $\Lambda_{-\ell}^k$

$$(5.19) \quad \square|_{\Lambda_{-\ell}^k} = \sum_{\substack{\alpha \in \Delta(w,a) \\ a-\ell < \alpha}} \underbrace{\mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}}_{\Lambda_{-\ell}^k} - \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ a-k-2\ell < m \leq a-k-\ell}} \underbrace{\mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}}_{\Lambda_{-\ell-m}^k} + \sum_{\substack{\alpha \in \Delta(w,a) \\ k < \alpha \leq a-\ell}} \underbrace{\epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^*}_{\Lambda_{k-a}^k}$$

The underbraces above indicate which $\Lambda_{-\bullet}^k$ the operator takes values in.

Bidegree $(2, k)$. Let

$$(5.20a) \quad M^k := \bigoplus_{c=k-a}^{a-1} M_c^k \quad \text{with} \quad M_c^k := \mathfrak{g}_{1,c} \otimes \mathfrak{g}_{1,k-c}, \quad 0 \leq k \leq 2a-1$$

be the component of $\mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^*$ of (Z_i, Z_w) -bidegree $(2, k)$. Note that $\mathcal{H}_{2,k}^1 = M^k \cap \mathcal{H}_2^1$, cf. (5.14). If $c < 0$ then $\mathfrak{g}_{1,c} = \{0\}$; if $k < c$, then $\mathfrak{g}_{1,k-c} = \{0\}$. Thus,

$$(5.20b) \quad M_c^k = \{0\} \quad \text{if either} \quad c < 0 \quad \text{or} \quad k < c.$$

By (3.2) the adjoint ∂^* vanishes on M^k . So, (5.16) yields

$$\square|_{M_c^k} = \sum_{\substack{\alpha \in \Delta(w,a) \\ c < \alpha}} \left(\partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial^* \right).$$

Applying Lemma 5.15, a computation similar to that for (5.19) yields

$$(5.21) \quad \square|_{M_c^k} = \sum_{\substack{\alpha \in \Delta(w,a) \\ c < \alpha}} \underbrace{\mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}}_{\text{valued in } M_c^k} - \sum_{\substack{\zeta_j \in \mathfrak{g}_{0,m} \\ 2c-k < m}} \underbrace{\mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}}_{\text{valued in } M_{c-m}^k}$$

Casimirs. Note that the $\sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}$ term of (5.19) and (5.21) satisfies

$$(5.22) \quad \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j} = \frac{1}{2} \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \left(\mathcal{L}_{\zeta_j} \circ \mathcal{L}_{\xi_j} - \mathcal{L}'_{\zeta_j} \circ \mathcal{L}'_{\xi_j} - \mathcal{L}''_{\zeta_j} \circ \mathcal{L}''_{\xi_j} \right).$$

The terms $\sum \mathcal{L} \circ \mathcal{L}$, $\sum \mathcal{L}' \circ \mathcal{L}'$ and $\sum \mathcal{L}'' \circ \mathcal{L}''$ appearing in the right-hand side of (5.22) are the $\mathfrak{g}_{0,0}$ -Casimirs on $\mathfrak{g}_w^\perp \otimes \mathfrak{n}_w^+$, \mathfrak{g}_w^\perp and \mathfrak{n}_w^+ , respectively.

Lemma 5.23. *Let $U_\beta \subset \mathfrak{g}_w^\perp$ and $U_\gamma \subset \mathfrak{n}_w^+$ be irreducible $\mathfrak{g}_{0,0}$ -modules of highest weights $\beta, \gamma \in \Delta(\mathfrak{g})$. Let $U_\pi \subset U_\beta \otimes U_\gamma$ be an irreducible $\mathfrak{g}_{0,0}$ -module of highest weight π . Then (5.22) acts on U_π by a scalar $c \leq (\beta, \gamma)$ with equality if and only if $\pi = \beta + \gamma$.*

Proof. Recall that the Casimir acts on an irreducible module of highest weight ν by the scalar $|\nu|^2 + 2(\nu, \rho_0)$, where $\rho_0 := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{0,0})} \alpha$.

The highest weight occurring in $U_\beta \otimes U_\gamma$ is $\beta + \gamma$. Because both π and $\beta + \gamma$ lie in the dominant Weyl chamber, we have $|\pi| \leq |\beta + \gamma|$. The weight ρ_0 lies in the interior of the dominant Weyl chamber, and so has the property that $(\gamma + \beta - \pi, \rho_0) \geq 0$, with equality if and only if $\pi = \gamma + \beta$. It follows that (5.22) acts by the scalar

$$\begin{aligned} c &= \frac{1}{2} (|\pi|^2 - |\beta|^2 - |\gamma|^2) + (\pi - \beta - \gamma, \rho_0) \\ &\stackrel{(\star)}{\leq} \frac{1}{2} (|\beta + \gamma|^2 - |\beta|^2 - |\gamma|^2) + (\pi - \beta - \gamma, \rho_0) \\ &= (\gamma, \beta) + (\pi - \beta - \gamma, \rho_0) \stackrel{(\dagger)}{\leq} (\gamma, \beta). \end{aligned}$$

Equality holds in (\star) and (\dagger) if and only if $\pi = \beta + \gamma$. \square

- Lemma 5.24.** (a) Let $\varsigma, \tau \in \Delta(\mathfrak{g})$. Then $L'_{E_\varsigma} \circ L'_{E_{-\varsigma}}$ acts on \mathfrak{g}_τ by a scalar $c \geq 0$. Moreover, $c = 0$ if and only if $\varsigma - \tau \notin \Delta(\mathfrak{g})$.
- (b) Assume $\varsigma \in \Delta(\mathfrak{g}_1)$ and $\tau \in \Delta(\mathfrak{g}_0)$. If both $\varsigma \pm \tau \in \Delta(\mathfrak{g})$, then $c = |\varsigma|^2$. If $\varsigma - \tau \in \Delta(\mathfrak{g})$ and $\varsigma + \tau \notin \Delta(\mathfrak{g})$, then $c = (\varsigma, \tau) > 0$.
- (c) Assume $\varsigma, \tau \in \Delta(\mathfrak{g}_1)$. If $\varsigma - \tau \in \Delta(\mathfrak{g})$, then $c = (\varsigma, \tau) > 0$. Otherwise $c = 0$.

Proof. The lemma is the specialization of a standard result [13, Corollary 2.37] in representation theory to the case that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. \square

5.3. Bidegree $(1, \mathbf{a} - 1)$. Recall the bi-graded decomposition (5.14). The computation that follows will determine necessary and sufficient conditions for $\mathcal{H}_{1, \mathbf{a}-1}^1 = \{0\}$, cf. Lemma 5.28.

Equations (5.19) and (5.22) yield

$$(5.25) \quad \square_{|\Lambda^{\mathbf{a}-1}} = \sum_{\alpha \in \Delta(w, \mathbf{a})} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} - \frac{1}{2} \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \left(\mathcal{L}_{\zeta_j} \circ \mathcal{L}_{\xi_j} - \mathcal{L}'_{\zeta_j} \circ \mathcal{L}'_{\xi_j} - \mathcal{L}''_{\zeta_j} \circ \mathcal{L}''_{\xi_j} \right).$$

The Laplacian acts on an irreducible $\mathfrak{g}_{0,0}$ -submodule $U \subset \Lambda^{\mathbf{a}-1}$ by a scalar, which is determined as follows. Recall from Section 3.2 that the irreducible $\mathfrak{g}_{0,0}$ -submodules of $\mathfrak{g}_{0,-1}$ are the $\mathfrak{g}_{0,-B}$ with $|B| = 1$. The highest weight of $\mathfrak{g}_{0,-B}$ is $-\beta = -\alpha_j$ for some $j \in J$. (Recall that α_j is a simple root of \mathfrak{g} .) Similarly, the irreducible $\mathfrak{g}_{0,0}$ -submodules of $\mathfrak{g}_{1,\mathbf{a}}$ are the $\mathfrak{g}_{1,C}$ with $|C| = \mathbf{a}$. Let $\gamma \in \Delta(w, \mathbf{a})$ be the highest weight of $\mathfrak{g}_{1,C}$. Let $U \subset \mathfrak{g}_{0,-B} \otimes \mathfrak{g}_{1,C}$ be an irreducible $\mathfrak{g}_{0,0}$ -submodule of highest weight π . From Lemmas 5.23 & 5.24 it follows that

$$(5.26) \quad \square \text{ acts on } U \text{ by a scalar } \stackrel{(\diamond)}{\geq} (\gamma, \beta) + \underbrace{\sum_{\alpha \pm \beta \in \Delta} |\alpha|^2 - \sum_{\substack{\alpha + \beta \in \Delta \\ \alpha - \beta \notin \Delta}} (\alpha, \beta)}_{\text{sums over } \alpha \in \Delta(w, \mathbf{a})} \stackrel{(\star)}{\geq} 0,$$

and equality holds at (\diamond) if and only if $\pi = \gamma - \beta$.

We now determine when this eigenvalue is zero (equivalently, when $\mathcal{H}_{1, \mathbf{a}-1}^1 \neq \{0\}$). There are three cases to consider:

- $(\gamma, \beta) = 0$: If equality holds at (\star) , then it must be the case that $[\mathfrak{g}_{1,\mathbf{a}}, \mathfrak{g}_{-\beta}] = \{0\}$. By Proposition 3.9, this implies that $\mathfrak{g}_{-\beta} \subset \mathfrak{g}_{0,-}$ stabilizes \mathbf{n}_w . This contradicts (3.8). Thus, strict inequality holds at (\star) , and \square acts on U by a positive scalar.
- $(\gamma, \beta) > 0$: Then \square acts on U by a scalar $\geq (\gamma, \beta) > 0$.
- $(\gamma, \beta) < 0$: Then \square acts on U by zero if and only if $\pi = \gamma - \beta$ and $\{\alpha \in \Delta(w, \mathbf{a}) \mid \alpha + \beta \in \Delta\} = \{\gamma\}$. The latter condition is equivalent to $[\mathfrak{g}_\beta, \mathfrak{g}_{1,\mathbf{a}}] = [\mathfrak{g}_\beta, \mathfrak{g}_\gamma] \neq \{0\}$. Recall that $(\gamma, \beta) < 0$ implies $\gamma - \beta \notin \Delta$, which is equivalent to $[\mathfrak{g}_\gamma, \mathfrak{g}_{-\beta}] = \{0\}$.

Definition 5.27. We say that H_1 is satisfied if there exist no irreducible $\mathfrak{g}_{0,0}$ -sub-modules $\mathfrak{g}_{0,-B} \subset \mathfrak{g}_{0,-1}$ and $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,\mathbf{a}}$ with highest weights $-\beta \in \Delta(\mathfrak{g}_{0,-B})$ and $\gamma \in \Delta(\mathfrak{g}_{1,C})$, respectively, such that the following three conditions hold

$$[\mathfrak{g}_{-\beta}, \mathfrak{g}_\gamma] = \{0\} \neq [\mathfrak{g}_\beta, \mathfrak{g}_\gamma] = [\mathfrak{g}_\beta, \mathfrak{g}_{1,\mathbf{a}}].$$

Let λ^k denote the $\mathcal{H}_{1,k}^1$ -valued component of λ , see (5.14). We have established the following.

Lemma 5.28. *The eigenvalues of the Laplacian on Λ^{a-1} are strictly positive if and only if H_1 is satisfied. Equivalently, $\mathcal{H}_{1,a-1}^1 = \{0\}$ if and only if H_1 is satisfied. In particular, if H_1 is satisfied, then λ^{a-1} vanishes on \mathcal{F}^0 , cf. (5.13).*

If H_1 fails for a pair $-\beta, \gamma$ then $\mathcal{H}_{1,a-1}^1$ contains an irreducible $\mathfrak{g}_{0,0}$ -module of highest weight $\gamma - \beta$.

5.4. Induction. In this section we prove the following.

Proposition 5.29. *If H_1 is satisfied, then λ (equivalently, $\vartheta_{0,-}$) is identically zero on \mathcal{F}^0 .*

We argue by induction. By Lemma 5.28, $\lambda^{>a-2}$ vanishes on \mathcal{F}^0 . Given any $k < a - 1$ with the property that $\lambda^{>k} = 0$ on \mathcal{F}^0 , we will show that $\lambda^{\geq k}$ vanishes on \mathcal{F}^0 . This will establish Proposition 5.29.

Write $\lambda^k = (\lambda_{-1}^k, \dots, \lambda_{k-a}^k)$, where $\lambda_{-\ell}^k$ denotes the $\Lambda_{-\ell}^k$ component of λ^k ; see (5.17), and recall that $\lambda_{-\ell}^k = 0$ if $k + \ell < 0$.

Lemma 5.30.

- (a) *Given $0 < \ell < a - k$ and $\zeta \in \mathfrak{g}_{0,a-k-\ell}$, we have $\mathcal{L}'_{\zeta}(\lambda_{k-a}^k) = -\mathcal{L}''_{\zeta}(\lambda_{-\ell}^k)$.*
- (b) *If $\lambda_{k-a}^k = 0$, then $\lambda^k = (\lambda_{-1}^k, \dots, \lambda_{k-a}^k) = 0$.*
- (c) *The component λ_{k-a}^k vanishes.*

Lemma 5.30 yields $\lambda^{\geq k} = 0$, completing the induction and establishing Proposition 5.29.

Proof of Lemma 5.30(a). Let $\lambda_{-\ell}$ denote the component of λ taking values in $\mathfrak{g}_{0,-\ell} \otimes \mathfrak{n}_w^*$, so that $\vartheta_{0,-\ell} = \lambda_{-\ell}(\vartheta_{\mathfrak{n}_w})$. Applying the Maurer-Cartan equation (4.4) to this equation yields

$$\begin{aligned}
 0 &\equiv d\lambda_{-\ell} \wedge \vartheta_{\mathfrak{n}_w} + \sum_{a=\ell}^a [\vartheta_{1,a-\ell}, \vartheta_{-1,-a}] + \sum_{m \geq \ell} [\vartheta_{0,m-\ell}, \lambda_{-m}(\vartheta_{\mathfrak{n}_w})] \\
 (5.31) \quad &\quad - \lambda_{-\ell}([\vartheta_{0,\geq 0}, \vartheta_{\mathfrak{n}_w}]) - \lambda_{-\ell}([\vartheta_{\mathfrak{n}_w}, \lambda(\vartheta_{\mathfrak{n}_w})]) \\
 &\quad + \frac{1}{2} \sum_{0 < m < \ell} [\lambda_{-m}(\vartheta_{\mathfrak{n}_w}), \lambda_{m-\ell}(\vartheta_{\mathfrak{n}_w})]
 \end{aligned}$$

The vanishing of $\lambda^{>k}$ on \mathcal{F}^0 is equivalent to

$$(5.32) \quad \lambda_{-\ell}(\mathfrak{g}_{-1,-a}) = \{0\} \quad \text{when} \quad 0 \leq a \leq a \quad \text{and} \quad k+1 \leq a-\ell \leq a-1.$$

Given $u \in \mathfrak{g}_{-1,-a}$ and $\zeta \in \mathfrak{g}_{0,m} \subset \mathfrak{g}_{0,>0}$, the coframing of Lemma 4.21 defines vector fields \hat{u} and $\hat{\zeta}$ on \mathcal{F}^0 by the conditions $\vartheta_{\mathfrak{g}_w}(\hat{u}) = u$ and $\vartheta_{\mathfrak{g}_w}(\hat{\zeta}) = \zeta$. Evaluating the 2-form (5.31) on $\hat{u} \wedge \hat{\zeta}$, and making use of (5.32) yields*

$$(5.33) \quad [\zeta, \lambda_{-\ell-m}(u)] = \lambda_{-\ell}([\zeta, u]),$$

for all $u \in \mathfrak{g}_{-1,-a} \subset \mathfrak{n}_w$, such that $k+1 \leq a-\ell \leq a-1$, and $\zeta \in \mathfrak{g}_{0,m} \subset \mathfrak{g}_{0,>0}$. Setting $a = a$ and $m = a - k - \ell$ yields $\lambda_{-\ell}([\zeta, u]) = [\zeta, \lambda_{k-a}^k(u)]$ for all $u \in \mathfrak{g}_{-1,-a}$ and $\zeta \in \mathfrak{g}_{0,m}$. Under the identification $\mathfrak{n}_w^* \simeq \mathfrak{n}_w^+$ this is equivalent to $\mathcal{L}'_{\zeta}(\lambda_{k-a}^k) = -\mathcal{L}''_{\zeta}(\lambda_{-\ell}^k)$ \square

Proof of Lemma 5.30(b). By (5.33) we have $[\zeta, \lambda_{k-a}^k(u)] = \lambda_{k-a+1}^k([\zeta, u])$, for all $\zeta \in \mathfrak{g}_{0,1}$ and $u \in \mathfrak{g}_{-1,-a}$. We will show that $\lambda_{k-a}^k = 0$ implies $\lambda_{k-a+1}^k = 0$. Part (b) will then follow from an inductive argument.

*In the context of Section 4.6, (5.33) is additional torsion in the system imposed by (5.32).

Assume $\lambda_{k-a}^k = 0$. Equivalently, $\lambda_{k-a}(u) = 0$ for all $u \in \mathfrak{g}_{-1,-a}$. Then $0 = [\zeta, \lambda_{k-a}(u)] = \lambda_{k-a+1}([\zeta, u])$ for all $\zeta \in \mathfrak{g}_{0,1}$. Lemma 3.14 implies $[\mathfrak{g}_{0,1}, \mathfrak{g}_{-1,-a}] = \mathfrak{g}_{-1,1-a}$. It follows that $\lambda_{k-a+1}(\mathfrak{g}_{-1,1-a}) = 0$. Equivalently, $\lambda_{k-a+1}^k = 0$. \square

Proof of Lemma 5.30(c). From (5.19), the component of $\square|_{\Lambda^k}$ taking values in Λ_{k-a}^k is

$$\square(\lambda_{-1}^k, \dots, \lambda_{k-a}^k)_{\Lambda_{k-a}^k} = \sum_{\substack{\alpha \in \Delta(w,a) \\ k < a}} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}(\lambda_{k-a}^k) - \sum_{\ell=1}^{a-k} \left(\sum_{\zeta_j \in \mathfrak{g}_{0,a-k-\ell}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}(\lambda_{-\ell}^k) \right).$$

An application of Lemma 5.30(a) to this expression yields

$$\begin{aligned} \square(\lambda_{-1}^k, \dots, \lambda_{k-a}^k)_{\Lambda_{k-a}^k} &= \sum_{\substack{\alpha \in \Delta(w,a) \\ k < a}} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}}(\lambda_{k-a}^k) - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}(\lambda_{k-a}^k) \\ (5.34) \quad &+ \sum_{\zeta_j \in \mathfrak{m}^{a-k-1}} \mathcal{L}'_{\xi_j} \circ \mathcal{L}'_{\zeta_j}(\lambda_{k-a}^k), \end{aligned}$$

where

$$\mathfrak{m}^{a-k-1} := \mathfrak{g}_{0,1} \oplus \dots \oplus \mathfrak{g}_{0,a-k-1}.$$

Define $\widehat{\square} : \Lambda_{k-a}^k \rightarrow \Lambda_{k-a}^k$ by

$$\widehat{\square} := \sum_{\substack{\alpha \in \Delta(w,a) \\ k < a}} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} + \sum_{\zeta_j \in \mathfrak{m}^{a-k-1}} \mathcal{L}'_{\xi_j} \circ \mathcal{L}'_{\zeta_j} - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}.$$

Then (5.34) reads $\square(\lambda_{-1}^k, \dots, \lambda_{k-a}^k)_{\Lambda_{k-a}^k} = \widehat{\square}(\lambda_{k-a}^k)$. The left-hand side must vanish, since λ^k takes values in $\mathcal{H}_{1,k}^1$. So it remains to show that $\ker \widehat{\square} = \{0\}$.

Recall that $\Lambda_{k-a}^k = \mathfrak{g}_{0,k-a} \otimes \mathfrak{g}_{1,a}$. Note that $\widehat{\square}$ is a $\mathfrak{g}_{0,0}$ -module map. Therefore Λ_{k-a}^k admits a $\mathfrak{g}_{0,0}$ -module decomposition into $\widehat{\square}$ -eigenspaces. The eigenvalues of $\widehat{\square}$ are computed as follows. Let $\mathfrak{g}_{0,-B} \subset \mathfrak{g}_{0,k-a}$ be an irreducible $\mathfrak{g}_{0,0}$ -submodule of highest weight $-\beta \in \Delta(\mathfrak{g}_{0,-B})$; let $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,a}$ be an irreducible module of highest weight $\gamma \in \Delta(\mathfrak{g}_{1,C})$. Let $U \subset \mathfrak{g}_{0,-B} \otimes \mathfrak{g}_{1,C} \subset \Lambda_{k-a}^k$ be an irreducible module of highest weight π . From Lemmas 5.23 & 5.24 we deduce that

$$(5.35) \quad \widehat{\square} \text{ acts on } U \text{ by a scalar } \stackrel{(*)}{\geq} (\gamma, \beta) + \underbrace{\sum_{\alpha \pm \beta \in \Delta} |\alpha|^2 - \sum_{\substack{\alpha + \beta \in \Delta \\ \alpha - \beta \notin \Delta}} (\alpha, \beta)}_{\text{sums over } \{\alpha \in \Delta(w,a) \mid k < a\}} \geq 0,$$

and equality holds at $(*)$ if and only if $\pi = \gamma - \beta$ and $[\mathfrak{m}^{a-k-1}, \mathfrak{g}_{-\beta}] = \{0\}$. The latter implies $[\mathfrak{g}_{0,1}, \mathfrak{g}_{-\beta}] = \{0\}$. Equivalently, $\mathfrak{g}_{-\beta} \not\subset [\mathfrak{g}_{0,-1}, \mathfrak{g}_{0,1-b}]$ where $b = \beta(Z_w) = a - k > 1$. On the other hand $\mathfrak{g}_{-\beta}$ is obtained from the highest root space by successive brackets with $\mathfrak{g}_{-\alpha_j}$, where α_j is a simple root. Since $-\beta$ is a highest $\mathfrak{g}_{0,0}$ -weight, this implies $\mathfrak{g}_{-\beta} \subset [\mathfrak{g}_{-1,0}, \mathfrak{g}_{1,-b}]$; but $\mathfrak{g}_{1,-b} = \{0\}$, yielding a contradiction. Thus, strict inequality holds at $(*)$, and $\widehat{\square}$ acts by a positive scalar. Thus $\ker \widehat{\square} = \{0\}$. \square

5.5. The finish. Assume that condition H_1 is satisfied. By Proposition 5.29 and Corollary 5.12 there exists $\mu : \mathcal{F}^0 \rightarrow \mathcal{H}_2^1$ such that $\vartheta_{1,<a} = \mu(\vartheta_{\mathfrak{n}_w^\perp})$.

Definition 5.36. We say that H_2 is satisfied if there exist no irreducible $\mathfrak{g}_{0,0}$ -sub-modules $\mathfrak{g}_{1,E} \subset \mathfrak{g}_{1,a-1}$ and $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,a}$ with highest weights $\varepsilon \in \Delta(\mathfrak{g}_{1,E})$ and $\gamma \in \Delta(\mathfrak{g}_{1,C})$, respectively, such that the following two conditions hold $\{0\} \neq [\mathfrak{g}_\varepsilon, \mathfrak{g}_{-\gamma}] = [\mathfrak{g}_\varepsilon, \mathfrak{g}_{-1,-a}]$.

We say that H_+ is satisfied if both H_1 (Definition 5.27) and H_2 are satisfied.

Proposition 5.37. *If H_+ is satisfied (Definition 5.36), then $\vartheta_{1,<a} = 0$ on \mathcal{F}^0 .*

Propositions 4.13, 5.29 and Proposition 5.37 establish the following.

Theorem 5.38. *If H_+ is satisfied (Definition 5.36), then the Schubert system \mathcal{B}_w is rigid. In particular, X_w is Schubert rigid.*

In Section 6 we identify those Schubert varieties X_w for which H_+ is satisfied. The remainder of this section is devoted to the proof, by induction, of Proposition 5.37. Let μ^k denote the component of μ taking values in M^k (cf. Section 5.2). We begin with $M^{2a-1} = \mathfrak{g}_{1,a-1} \otimes \mathfrak{g}_{1,a}$.

Lemma 5.39. *Assume H_1 is satisfied. If H_2 is satisfied, then the eigenvalues of the Laplacian on M^{2a-1} are strictly positive. In particular, $\mathcal{H}_{2,2a-1}^1 = \{0\}$ and μ^{2a-1} vanishes.*

If H_2 fails for a pair β, γ , then $\mathcal{H}_{2,2a-1}^1$ contains an irreducible $\mathfrak{g}_{0,0}$ -module of highest weight $\gamma + \beta$.

Proof. It suffices to show that the eigenvalues of \square on $M_{2a-1} = \mathfrak{g}_{1,a-1} \otimes \mathfrak{g}_{1,a}$ are strictly positive. The computations involved are similar to those of Section 5.3. Let $\varepsilon \in \Delta(\mathfrak{g}_{1,a-1})$ be a highest weight of an irreducible $\mathfrak{g}_{1,E} \subset \mathfrak{g}_{1,a-1}$, and let $\gamma \in \Delta(\mathfrak{g}_{1,a})$ be a highest weight of an irreducible $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,a}$. Fix an irreducible $\mathfrak{g}_{0,0}$ -module $U \subset \mathfrak{g}_{1,E} \otimes \mathfrak{g}_{1,C}$ of highest weight π . From (5.21), Lemmas 5.23 & 5.24(c) we deduce

$$\square \text{ acts on } U \text{ by a scalar } \geq -(\gamma, \varepsilon) + \sum_{\substack{\alpha \in \Delta(w,a) \\ \alpha - \varepsilon \in \Delta}} (\alpha, \varepsilon) \geq 0.$$

This sum may vanish under two conditions. Both require $\pi = \varepsilon + \gamma$. First, it may be the case that $[\mathfrak{g}_\varepsilon, \mathfrak{g}_{-1,-a}] = \{0\} \subset \mathfrak{g}_{0,-1}$. This forces $\mathfrak{g}_\varepsilon \notin [\mathfrak{g}_{0,-1}, \mathfrak{g}_{1,a}]$, which contradicts Lemma 3.14. Second, it may be the case that $[\mathfrak{g}_\varepsilon, \mathfrak{g}_{-1,-a}] = [\mathfrak{g}_\varepsilon, \mathfrak{g}_{-\gamma}] \neq \{0\}$. This is precisely the case that the condition H_2 fails. The lemma is established. \square

Assume that H_+ is satisfied, and that there exists $0 \leq k \leq 2a-2$ such that $\mu^{>k}$ vanishes on \mathcal{F}^0 . Lemma 5.39 assures us that the inductive hypothesis holds for $k = 2a-2$. We will show that μ^k also vanishes on \mathcal{F}^0 . Let μ_c^k denote the component of μ taking value in M_c^k , see (5.20) and recall that $\mu_c^k = 0$ if either $c < 0$ or $k < c$.

Lemma 5.40.

- (a) *Given $c > k - a$ and $\zeta \in \mathfrak{g}_{0,a-k+c}$, we have $\mathcal{L}_\zeta''(\mu_c^k) = -\mathcal{L}_\zeta'(\mu_{k-a}^k)$.*
- (b) *If $\mu_{k-a}^k = 0$, then $\mu^k = (\mu_{k-a}^k, \dots, \mu_{a-1}^k) = 0$.*
- (c) *The component μ_{k-a}^k vanishes.*

Lemma 5.40 yields $\mu^{\geq k} = 0$, completing the induction and establishing Proposition 5.37.

Proof of Lemma 5.40(a). Let μ_c denote the component of μ taking values in $\mathfrak{g}_{1,c} \otimes \mathfrak{n}_w^*$. The vanishing of $\mu^{>k}$ is equivalent to $\mu_c(u) = 0$ for all $u \in \mathfrak{g}_{-1,-a}$ with $c+a > k$. An application of the Maurer-Cartan equation (4.4) to $\vartheta_{1,c} = \mu_c(\vartheta_{\mathfrak{n}_w})$ yields

$$(5.41) \quad \frac{1}{2} d\mu_c \wedge \vartheta_{\mathfrak{n}_w} = \mu_c([\vartheta_{0,\geq 0}, \vartheta_{\mathfrak{n}_w}]) - [\vartheta_{0,\geq 0}, \vartheta_{1,<a}]_{1,c}.$$

In particular, if $\zeta \in \mathfrak{g}_{0,m} \subset \mathfrak{g}_{0,>0}$ and $u \in \mathfrak{g}_{-1,-a} \subset \mathfrak{n}_w$ with $a+c > k$, then evaluating (5.41) on $\hat{\zeta} \wedge \hat{u}$ yields $0 = \mu_c([\zeta, u]) - [\zeta, \mu_{c-m}(u)]$. Set $a = \mathfrak{a}$ and $m = \mathfrak{a} - k + c$. Under the identification $\mathfrak{n}_w^* \simeq \mathfrak{n}_w^+$, this yields $\mathcal{L}'_{\zeta}(\mu_c^k) = -\mathcal{L}''_{\zeta}(\mu_{c+p}^k)$. \square

Proof of Lemma 5.40(b). The proof, which is identical to that of Lemma 5.30(b), is left to the reader. \square

Proof of Lemma 5.40(c). From (5.21), the component of $\square|_{M^k}$ taking value in $M_{k-\mathfrak{a}}^k$ is

$$(5.42) \quad \begin{aligned} \square(\mu_{k-\mathfrak{a}}^k, \dots, \mu_{\mathfrak{a}-1}^k)_{M_{k-\mathfrak{a}}^k} &= \sum_{\substack{\alpha \in \Delta(w,a) \\ k-\mathfrak{a} < a}} \mathcal{L}'_{E_{\alpha}} \circ \mathcal{L}'_{E_{-\alpha}}(\mu_{k-\mathfrak{a}}^k) - \sum_{\substack{k-\mathfrak{a} \leq c < a \\ \zeta_j \in \mathfrak{g}_{\mathfrak{a}-k+c}}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}(\mu_c^k) \\ &\stackrel{(\dagger)}{=} \sum_{\substack{\alpha \in \Delta(w,a) \\ k-\mathfrak{a} < a}} \mathcal{L}'_{E_{\alpha}} \circ \mathcal{L}'_{E_{-\alpha}}(\mu_{k-\mathfrak{a}}^k) + \sum_{\zeta_j \in \mathfrak{m}^{2\mathfrak{a}-1-k}} \mathcal{L}'_{\xi_j} \circ \mathcal{L}'_{\zeta_j}(\mu_{k-\mathfrak{a}}^k) \\ &\quad - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}(\mu_{k-\mathfrak{a}}^k); \end{aligned}$$

the equality (\dagger) is a consequence of Lemma 5.40(a).

Define $\tilde{\square} : M_{k-\mathfrak{a}}^k \rightarrow M_{k-\mathfrak{a}}^k$ by

$$\tilde{\square} := \sum_{\substack{\alpha \in \Delta(w,a) \\ k-\mathfrak{a} < a}} \mathcal{L}'_{E_{\alpha}} \circ \mathcal{L}'_{E_{-\alpha}} + \sum_{\zeta_j \in \mathfrak{m}^{2\mathfrak{a}-1-k}} \mathcal{L}'_{\zeta_j} \circ \mathcal{L}'_{\xi_j} - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}.$$

Then (5.42) reads $\square(\mu_{k-\mathfrak{a}}^k, \dots, \mu_{\mathfrak{a}-1}^k)_{M_{k-\mathfrak{a}}^k} = \tilde{\square}(\mu_{k-\mathfrak{a}}^k)$. Since μ^k is $\mathcal{H}_{2,k}^1$ -valued, the left-hand side of this equation vanishes. To complete the proof, it suffices to show that $\ker \tilde{\square} = \{0\}$.

Note that $\tilde{\square}$ is a $\mathfrak{g}_{0,0}$ -module map. Therefore $M_{k-\mathfrak{a}}^k$ admits a $\mathfrak{g}_{0,0}$ -module decomposition into $\tilde{\square}$ -eigenspaces. To see that $\ker \tilde{\square}$ is trivial, let $\beta \in \Delta(\mathfrak{g}_{1,k-\mathfrak{a}})$ be a highest weight of an irreducible $\mathfrak{g}_{1,B} \subset \mathfrak{g}_{1,k-\mathfrak{a}}$, and let $\gamma \in \Delta(\mathfrak{g}_{1,\mathfrak{a}})$ be a highest weight of an irreducible $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,\mathfrak{a}}$. Fix an irreducible $\mathfrak{g}_{0,0}$ -module $U \subset \mathfrak{g}_{1,B} \otimes \mathfrak{g}_{1,C}$ of highest weight π . From (5.21), Lemmas 5.23 and 5.24.c we deduce

$$\tilde{\square} \text{ acts on } U \text{ by a scalar } \stackrel{(*)}{\geq} -(\gamma, \beta) + \sum_{\substack{\alpha \in \Delta(w,a) \\ k-\mathfrak{a} < a}} (\alpha, \beta) \geq 0,$$

and equality holds in $(*)$ if and only if $[\mathfrak{m}^{2\mathfrak{a}-1-k}, \mathfrak{g}_{\beta}] = \{0\}$. Since $\mathfrak{g}_{0,1} \subset \mathfrak{m}^{2\mathfrak{a}-1-k}$, equality at $(*)$ implies $[\mathfrak{g}_{0,1}, \mathfrak{g}_{\beta}] = \{0\}$. This contradicts Lemma 3.14. Thus, strict inequality holds at $(*)$, and $\tilde{\square}$ acts on U by a positive scalar. \square

6. SCHUBERT RIGIDITY

By Theorem 5.38, the Schubert system \mathcal{B}_w is rigid when H_+ is satisfied (Definition 5.36). In this section we will prove Theorem 6.1. Throughout the section

$$J = \{j_1, \dots, j_p\}$$

is the index set of Section 3.2, ordered so that $j_1 < \dots < j_p$; for convenience we set

$$j_0 := 0 \quad \text{and} \quad j_{p+1} := 1 + \max\{1, \dots, \hat{i}, \dots, n\}.$$

We assume that X_w is proper, so that $J \neq \emptyset$ (Remark 3.6). The admissible pairs (J, a) are listed in Corollary 3.17.

Theorem 6.1. *The Schubert varieties satisfying H_+ are listed below.*

- (a) In A_n/P_i with $1 < i < n$. Define $q \in \mathbb{Z}$ by $j_q < i < j_{q+1}$.
- ♠ : $(p, q) = (2a + 2, a + 1)$ with $1 < j_\ell - j_{\ell-1}$, for all $1 < \ell \leq p$, and $1 < i - j_q, j_{q+1} - i$;
 - ♡ : $(p, q) = (2a + 1, a + 1)$ with $1 < j_\ell - j_{\ell-1}$, for all $1 < \ell \leq p + 1$, and $1 < j_{q+1} - i$;
 - ◇ : $(p, q) = (2a + 1, a)$ with $1 < j_\ell - j_{\ell-1}$, for all $1 \leq \ell \leq p$, and $1 < i - j_q$;
 - ♣ : $(p, q) = (2a, a)$ with $1 < j_\ell - j_{\ell-1}$ for all $1 \leq \ell \leq p + 1$.
- (b) In D_n/P_1 : the X_w with $a = 0$ and $J = \{n - 1\}$ or $J = \{n\}$.
- (c) In C_n/P_n :
- (i) when $p = a$, any J with $1 < j_\ell - j_{\ell-1}$ for all $1 \leq \ell \leq p$;
 - (ii) when $p = a + 1$, any J with $1 < j_\ell - j_{\ell-1}$ for all $2 \leq \ell \leq p + 1$.
- (d) In D_n/P_n , set $s_0 = \lceil \frac{p+1}{2} \rceil$ and $s_1 = \lceil \frac{p}{2} \rceil$:
- (i) Either $p = a$ and $n - 1 \notin J$, or $p = a + 1$ and $n - 1 \in J$; in both cases $1 < j_\ell - j_{\ell-1}$ for all $1 \leq \ell \leq p$ and $2 < j_s - j_{s-1}$ with $s = \lceil \frac{p+1}{2} \rceil$.
 - (ii) $p = a + 1$ and $n - 1 \notin J$ with $1 < j_\ell - j_{\ell-1}$ for all $2 \leq \ell \leq p$, and $2 < j_{s+1} - j_s$ with $s = \lceil \frac{p}{2} \rceil$.
- (e) In $E_6/P_1 \simeq E_6/P_6$ and E_7/P_7 : see Tables 4 & 5.[†]

TABLE 4. Proper $X_w \subset E_6/P_6$ satisfying H_+ .

w	a	J	$\dim X_w$	a^*	J^*
(6542)	0	$\{3\}$	4	1	$\{3\}$
(65431)	0	$\{2\}$	5	1	$\{5\}$
(65432413)	1	$\{4\}$	8	1	$\{4\}$
(65432456)	0	$\{1\}$	8	0	$\{1\}$
(65432451342)	1	$\{5\}$	11	0	$\{2\}$
(654324561345)	1	$\{3\}$	12	0	$\{3\}$

[†]The first column of Tables 4 & 5 expresses w as a composition of reflections associated to simple roots (acting on the left): for example, (6542) denotes $\sigma_6\sigma_5\sigma_4\sigma_2 \in W^p$, where $\sigma_i \in W$ is the reflection associated to the simple root α_i . The values a^* and J^* are given Lemma 3.22 and Proposition 3.24, respectively.

TABLE 5. Proper $X_w \subset E_7/P_7$ satisfying H_+ .

w	a	J	$\dim X_w$	a^*	J^*
(76542)	0	$\{3\}$	5	2	$\{5\}$
(765431)	0	$\{2\}$	6	1	$\{2\}$
(765432413)	1	$\{4\}$	9	2	$\{4\}$
(7654324567)	0	$\{1\}$	10	1	$\{6\}$
(765432451342)	1	$\{5\}$	12	1	$\{3\}$
(7654324561345)	2	$\{3, 6\}$	13	2	$\{1, 5\}$
(76543245671342)	2	$\{1, 5\}$	14	2	$\{3, 6\}$
(765432456713456)	1	$\{3\}$	15	1	$\{5\}$
(76543245613452431)	1	$\{6\}$	17	0	$\{1\}$
(765432456713456245)	2	$\{4\}$	18	1	$\{4\}$
(765432456713456245342)	1	$\{2\}$	21	0	$\{2\}$
(7654324567134562453413)	2	$\{5\}$	22	0	$\{3\}$

Remarks. By Theorem 5.38 the varieties listed in Theorem 6.1 are Schubert rigid. They are the Schubert varieties for which there exist first-order obstructions to the existence of nontrivial integral varieties of the Schubert system. In particular, those varieties not listed need not be flexible: there may exist higher-order obstructions.

By Proposition 7.3, a Schubert variety is Schubert rigid if and only if its conjugate is Schubert rigid. Making use of (3.21), we see that the list of Schubert varieties satisfying H_+ is also closed under conjugation.

By Corollary 7.2, $[X_{w^*}]$ is the Poincaré dual of $[X_w]$.

Corollary 6.2. *A Schubert variety satisfies H_+ if and only if its Poincaré dual does.*

Proof. It is clear from Tables 4 & 5 that the corollary holds for the exceptional CHSS. To prove the corollary for the classical CHSS we apply Proposition 3.24 and Remark 3.23 to rephrase Theorem 6.1 in Table 6.

We say that a subset S of the Dynkin diagram $\delta_{\mathfrak{g}}$ is *orthogonal* if distinct pairs $j, k \in S$ are not connected. Equivalently, the corresponding simple roots α_j and α_k are orthogonal with respect to the Killing form. Additionally, define $S_{<i} = \{j \in S \mid j < i\}$. The sets $S_{\leq i}$, $S_{>i}$ and $S_{\geq i}$ are defined analogously. Let

$$D := J \cup \{i\} \quad \text{and} \quad D^* := J^* \cup \{i\}$$

The corollary follows from inspection of Table 6, which lists the proper Schubert varieties in the classical CHSS that satisfy H_+ . \square

Remark 6.3. Applying Proposition 3.30 and Table 3 to Theorem 6.1(a) yields a partition description of the proper $X_\pi \subset \text{Gr}(i, n+1)$ satisfying H_+ . They are precisely those partitions $\pi = (p_1^{q_1}, \dots, p_r^{q_r})$ satisfying

$$1 < q_\ell, \quad q'_\ell, \quad \text{for all } 2 \leq \ell \leq r, \quad \text{and}$$

TABLE 6. The proper Schubert varieties in the classical CHSS satisfying H_+

A_n/P_i $1 < i < n$	♠	D orthogonal	
	♡	$D_{<i}, D_{\geq i}, D_{\geq i}^*$ orthogonal	
	◇	$D_{>i}, D_{\leq i}, D_{\leq i}^*$ orthogonal	
	♣	D^* orthogonal	
D_n/P_1	a = 0 and $J = \{n - 1\}$ or $\{n\}$		
C_n/P_n	D^* orthogonal, p = a		
	D orthogonal, p = a + 1		
D_n/P_n	J orthogonal	$p = a \ \& \ n - 1 \notin J$	$1 \notin J \ \& \ 2 < j_s - j_{s-1}, s = \lceil \frac{p+1}{2} \rceil$
		$p = a + 1 \ \& \ n - 1 \in J$	
		$p = a + 1 \ \& \ n - 1 \notin J$ with $2 < j_{s+1} - j_s, s = \lceil \frac{p}{2} \rceil$	

$$\begin{aligned}
1 < q'_1 \text{ when } \pi \in \heartsuit; & \quad 1 < q_1, q'_1 \text{ when } \pi \in \clubsuit; \\
1 < q_1 \text{ when } \pi \in \diamond; & \quad \text{no additional constraints when } \pi \in \spadesuit.
\end{aligned}$$

Outline of proof of Theorem 6.1. Sections 6.1 & 6.2 identify the Schubert varieties of the quadric hypersurface that satisfy H_+ yielding Theorem 6.1(b). Parts (c) & (d) are proved in Sections 6.3 & 6.4, respectively. The reader who works through these arguments will be able to prove (a) as an exercise. We used LiE [16] to verify Theorem 6.1(e). \square

Throughout the remainder of Section 6 the notation γ will denote a highest root of $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,a}$; $-\beta = -\alpha_j$ will denote a highest root of $\mathfrak{g}_{0,B} \subset \mathfrak{g}_{0,-1}$, $j \in J$; and ε will denote a highest root of $\mathfrak{g}_{1,E} \subset \mathfrak{g}_{1,a-1}$.

6.1. Quadric hypersurface $Q^{2n-1} = B_n/P_1$. By Corollary 3.17, we have $J = \{j\}$ and $a \in \{0, 1\}$. We begin with the case that $a = 1$. In this case $\mathfrak{g}_{1,1} = \mathfrak{g}_{1,C}$ is an irreducible $\mathfrak{g}_{0,0}$ -module with $C = (1)$; cf. the proof of Lemma 3.15. We have

$$\begin{aligned}
\gamma &= \alpha_1 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n), \\
\Delta(\mathfrak{g}_{1,1}) &= \{\alpha_1 + \cdots + \alpha_k, \alpha_1 + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_n) \mid k \geq j\}.
\end{aligned}$$

Condition H_1 fails for $\beta = \alpha_j$.

Next assume that $a = 0$. Then $\mathfrak{g}_{1,0} = \mathfrak{g}_{1,C}$ is again irreducible with $C = (0)$, and

$$(6.4) \quad \gamma = \alpha_1 + \cdots + \alpha_{j-1} \quad \text{and} \quad \Delta(\mathfrak{g}_{1,0}) = \{\alpha_1 + \cdots + \alpha_k \mid k < j\}.$$

The condition H_1 fails for $\beta = \alpha_j$.

We conclude that none of the proper Schubert varieties $X_w \subset Q$ satisfy H_+ .

6.2. Quadric hypersurface $Q^{2n-2} = D_n/P_1$. There are four cases to consider in Corollary 3.17. **(1)** Begin with $a = 0$ and $J = \{j\}$. Suppose that $j < n-1$. Then γ and $\Delta(\mathfrak{g}_{1,0})$ are given by (6.4). The condition H_1 fails for $\beta = \alpha_j$.

Now suppose $j \in \{n-1, n\}$. The two cases are symmetric; so without loss of generality, $j = n$. Then

$$\gamma = \alpha_1 + \cdots + \alpha_{n-1}, \quad \Delta(\mathfrak{g}_{1,0}) = \{\alpha_1 + \cdots + \alpha_j \mid j < n\}.$$

Condition H_1 is satisfied. Condition H_2 is vacuous as $\mathbf{a} = 0$.

(2) In the case $\mathbf{a} = 0$ and $J = \{n-1, n\}$, γ and $\Delta(\mathfrak{g}_{1,0})$ are given by (6.4) with j replaced by $n-1$. The condition H_1 fails for both $\beta = \alpha_{n-1}, \alpha_n$.

(3) Next suppose that $\mathbf{a} = 1$ and $J = \{j < n-1\}$. Then

$$\begin{aligned} \gamma &= \alpha_1 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \\ \Delta(\mathfrak{g}_{1,1}) &= \{ \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n, \alpha_1 + \cdots + \alpha_k, \\ &\quad \alpha_1 + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{n-1}) + \alpha_{n-1} + \alpha_n \mid k \geq j \}. \end{aligned}$$

The condition H_1 fails for $\beta = \alpha_j$.

(4) If $\mathbf{a} = 1$ and $J = \{n-1, n\}$, then $\mathfrak{g}_{1,1} = \mathfrak{g}_{1,(1,0)} \oplus \mathfrak{g}_{1,(0,1)} = \mathfrak{g}_{\gamma_1} \oplus \mathfrak{g}_{\gamma_2}$ where

$$\gamma_1 = \alpha_1 + \cdots + \alpha_{n-1} \quad \text{and} \quad \gamma_2 = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n.$$

The condition H_1 fails for the pairs $(\gamma_1, \beta = \alpha_n)$ and $(\gamma_2, \beta = \alpha_{n-1})$.

6.3. Lagrangian grassmannian C_n/P_n . By Corollary 3.17, $\mathbf{a} \leq |J| \leq \mathbf{a} + 1$.

Proposition. *The proper Schubert varieties $X_w \subset C_n/P_n$ satisfying H_+ are*

- (i) $|J| = \mathbf{a}$ with $1 < j_1$ and $1 < j_\ell - j_{\ell-1}$, for all $1 < \ell \leq \mathbf{a}$; and
- (ii) $|J| = \mathbf{a} + 1$ with $1 < j_\ell - j_{\ell-1}$, for all $1 < \ell \leq \mathbf{a} + 1$, and $j_{\mathbf{a}+1} < n-1$.

Proof of (i). The proposition is proved in two claims. We first warm-up with a simple case, and then prove the general case.

Claim. The Schubert variety X_w associated to $J = \{j\}$ with $\mathbf{a} = 1$ satisfies H_+ if and only if $1 < j$.

The $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{1,\mathbf{a}} = \mathfrak{g}_{1,C}$ is irreducible with $C = (1)$, highest weight $\gamma = \alpha_1 + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{n-1}) + \alpha_n$ and

$$\Delta(\mathfrak{g}_{1,\mathbf{a}}) = \{ \alpha_k + \cdots + \alpha_n, \alpha_k + \cdots + \alpha_{\ell-1} + 2(\alpha_\ell + \cdots + \alpha_{n-1}) + \alpha_n \mid k \leq j < \ell \}.$$

Condition H_1 is always satisfied. The $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{1,\mathbf{a}-1} = \mathfrak{g}_{1,0} = \mathfrak{g}_{1,E}$ is irreducible with $E = (0)$ and highest weight $\varepsilon = 2(\alpha_{j+1} + \cdots + \alpha_{n-1}) + \alpha_n$. Condition H_2 is satisfied if and only if $j > 1$. This establishes the claim.

Claim. The Schubert subvarieties $X_w \subset C_n/P_n$ with $\mathbf{p} = \mathbf{a}$ satisfy H_+ if and only if $1 < j_\ell - j_{\ell-1}$ for all $2 \leq \ell \leq \mathbf{a}$, and $1 < j_1$.

We have

$$\mathfrak{g}_{1,\mathbf{a}} = \bigoplus_{q=0}^{\lfloor \frac{\mathbf{a}}{2} \rfloor} \mathfrak{g}_{1,C_q} \quad \text{where} \quad C_q = (0^q, 1^{\mathbf{a}-2q}, 2^q).$$

Let γ_q denote the highest weight of \mathfrak{g}_{1,C_q} . Then

$$\begin{aligned} \gamma_q &= \alpha_{j_q+1} + \cdots + \alpha_{j_{\mathbf{a}-q}} + 2(\alpha_{j_{\mathbf{a}-q}+1} + \cdots + \alpha_{n-1}) + \alpha_n \\ \Delta(\mathfrak{g}_{1,C_q}) &= \{ \alpha_j + \cdots + \alpha_{k-1} + 2(\alpha_k + \cdots + \alpha_{n-1}) + \alpha_n \mid j_q < j \leq j_{q+1}, \\ &\quad j_{\mathbf{a}-q} < k \leq j_{\mathbf{a}-q+1} \}. \end{aligned}$$

Excluding the case that $\mathbf{a} = 2r + 1$ is odd and $q = r$, the conditions $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{\gamma_q}] = \{0\} \neq [\mathfrak{g}_\beta, \mathfrak{g}_{\gamma_q}]$ will hold if and only if $\beta = \alpha_{j_q}$ or $\beta = \alpha_{j_{\mathbf{a}-q}}$. Observe that $[\mathfrak{g}_{\alpha_{j_q}}, \mathfrak{g}_{1,\mathbf{a}}] =$

$[\mathfrak{g}_{\alpha_{j_q}}, \mathfrak{g}_{1,C_q}]$ and $[\mathfrak{g}_{\alpha_{j_{a-q}}}, \mathfrak{g}_{1,a}] = [\mathfrak{g}_{\alpha_{j_{a-q}}}, \mathfrak{g}_{1,C_q}]$. We have $[\mathfrak{g}_{\alpha_{j_q}}, \mathfrak{g}_{1,C_q}] = [\mathfrak{g}_{\alpha_{j_q}}, \mathfrak{g}_{\gamma_q}]$ if and only if $j_{a-q}+1 = j_{a-q+1}$. Similarly, $[\mathfrak{g}_{\alpha_{j_{a-q}}}, \mathfrak{g}_{1,C_q}] = [\mathfrak{g}_{\alpha_{j_{a-q}}}, \mathfrak{g}_{\gamma_q}]$ if and only if $j_q+1 = j_{q+1}$. In summary, H_1 is satisfied if and only if $1 < j_1$ and $1 < j_{q+1} - j_q$ (with the caveat that the last inequality is not required for $\ell = r$ when $a = 2r + 1$.)

To address the condition H_2 note that

$$\mathfrak{g}_{1,a-1} = \bigoplus_{r=0}^{\lfloor \frac{a-1}{2} \rfloor} \mathfrak{g}_{1,E_r} \quad \text{where} \quad E_r = (0^{r+1}, 1^{a-1-2r}, 2^r).$$

The highest weight of \mathfrak{g}_{1,E_r} is

$$\varepsilon_r = \alpha_{j_{r+1}+1} + \cdots + \alpha_{j_{a-r}} + 2(\alpha_{j_{a-r}+1} + \cdots + \alpha_{n-1}) + \alpha_n.$$

We have $[\mathfrak{g}_{-\varepsilon_r}, \mathfrak{g}_{\gamma_q}] \neq \{0\}$ if only if $r = q$ or $r = q - 1$. In particular, $[\mathfrak{g}_{-\varepsilon_r}, \mathfrak{g}_{1,a}] \supset \mathfrak{g}_{\gamma_r-\varepsilon_r} \oplus \mathfrak{g}_{\gamma_{r+1}-\varepsilon_r}$. Both summands are nonzero as long as $a - 1 \neq 2r$. (In this case there is no C_{r+1} .) So H_2 may only fail for $a = 2r + 1$ and pairs of the form $(\varepsilon_r, \gamma_r)$. The condition H_2 is satisfied when $1 < j_{r+1} - j_r$. This completes the proof of the claim. \square

Proof of (ii). The proof is given in a series of three claims. The first two are warm-ups, the third establishes the general result.

Claim. The Schubert variety associated to $a = 0$ and $J = \{j\}$ satisfies H_+ if and only if $j < n - 1$.

We have $\mathfrak{g}_{1,a} = \mathfrak{g}_{1,C}$ where $C = (0)$. The highest weight of $\mathfrak{g}_{1,C}$ is the root $\gamma = 2(\alpha_{j_1+1} + \cdots + \alpha_{n-1}) + \alpha_n$, and

$$\begin{aligned} \Delta(\mathfrak{g}_{1,a}) &= \{ \alpha_k + \cdots + \alpha_n, \alpha_k + \cdots + \alpha_{\ell-1} + 2(\alpha_\ell + \cdots + \alpha_{n-1}) + \alpha_n, \\ &\quad 2(\alpha_\ell + \cdots + \alpha_{n-1}) + \alpha_n \mid j < k, \ell \}. \end{aligned}$$

Condition H_1 will fail with $\beta = \alpha_j$ only if $j = n - 1$. Since $\mathfrak{g}_{1,a-1} = \{0\}$ condition H_2 is vacuous. The claim follows.

Claim. The Schubert variety X_w associated to $J = \{j_1, j_2\}$ with $a = 1$ satisfies H_+ if and only if $1 < j_2 - j_1$ and $j_2 < n - 1$.

The module $\mathfrak{g}_{1,a} = \mathfrak{g}_{1,C}$ is irreducible with $C = (0, 1)$ and

$$\begin{aligned} \gamma &= \alpha_{j_1+1} + \cdots + \alpha_{j_2} + 2(\alpha_{j_2+1} + \cdots + \alpha_{n-1}) + \alpha_n, \\ \Delta(\mathfrak{g}_{1,a}) &= \{ \alpha_k + \cdots + \alpha_n, \\ &\quad \alpha_k + \cdots + \alpha_{\ell-1} + 2(\alpha_\ell + \cdots + \alpha_{n-1}) + \alpha_n \mid j_1 < k \leq j_2 < \ell \}. \end{aligned}$$

If $\beta = \alpha_{j_1}$, then H_1 will fail if and only if $j_2 = n - 1$. The module $\mathfrak{g}_{1,a-1} = \mathfrak{g}_{1,E}$ is irreducible with $E = (0, 0)$ and highest weight $\varepsilon = 2(\alpha_{j_2+1} + \cdots + \alpha_{n-1}) + \alpha_n$. Condition H_2 will be satisfied if and only if $j_2 - j_1 > 1$. This proves the claim.

Claim. The Schubert subvarieties $X_w \subset C_n/P_n$ with $\mathbf{p} = \mathbf{a} + 1$ satisfy H_+ if and only if satisfying $1 < j_\ell - j_{\ell-1}$ for all $2 \leq \ell \leq a + 1$, and $j_{a+1} < n - 1$.

The proof of the claim, which is very similar to the proof of general case of (i), is left to the reader. \square

6.4. Spinor variety D_n/P_n .

Proposition. (a) If $a = 0$, then H_+ is satisfied if and only if $J \neq \{n - 2\}$.

(b) If $0 < a = 2r - 1$, $2r$ and $n - 1 \notin J$, then H_+ is satisfied if and only if $j_\ell - j_{\ell-1} > 1$, for all $p - a < \ell \leq p$, and $j_{p-r+1} - j_{p-r} > 2$.

(c) If $0 < a = 2r$, $2r + 1$ and $n - 1 \in J$, then H_+ is satisfied if and only if $|J| = a + 1$ and $j_\ell - j_{\ell-1} > 1$, for all $1 \leq \ell \leq p$, and $j_{p-r} - j_{p-r-1} > 2$.

Proof. Review Corollary 3.17 and Remark 3.16. The analysis proceeds as in Section 6.3. We leave it to the reader to verify the following.

(A) Assume that $n - 1 \notin J$. Then $a \leq p \leq a + 1$.

(A.1) Suppose that $a = 0$. Then $J = \{j\}$ and H_1 is satisfied if and only if $j \neq n - 2$. The condition H_2 is vacuous.

(A.2) Suppose that $a = 2r - 1$ or $a = 2r$, with $r > 0$. Then H_1 is satisfied if and only if $j_\ell - j_{\ell-1} > 1$, for all $p - a < \ell \leq p$, and $j_{p-r+1} - j_{p-r} > 2$. In this case H_2 is also satisfied.

(B) Suppose that $n - 1 \in J$. Then $a + 1 \leq |J| = p \leq a + 2$. If $p = a + 2$, then H_1 fails for $\beta = \alpha_{j_1}$ and γ the highest weight of $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,a}$ with $C = (0, 1^a, 0)$. So we assume $p = a + 1$.

(B.1) If $a = 0$, then H_1 is satisfied and H_2 is vacuous.

(B.2) Suppose that $0 < a = 2r$, $2r + 1$. Then H_1 is satisfied if and only if $j_\ell - j_{\ell-1} > 1$, for all $1 \leq \ell \leq p$, and $j_{p-r} - j_{p-r-1} > 2$. In this case H_2 is also satisfied. \square

7. THE SCHUR SYSTEM \mathcal{R}_w

7.1. Definition. Define $R_w = \mathbb{P}\mathbf{I}_w \cap \text{Gr}(|w|, \mathfrak{g}_{-1})$. The *Schur system* $\mathcal{R}_w \subset \text{Gr}(|w|, TX)$ is defined by $\mathcal{R}_{w,z} = g_* R_w$, $z = gP$, $g \in G$. A $|w|$ -dimensional complex submanifold $M \subset X$ is an *integral manifold* of \mathcal{R}_w if $TM \subset \mathcal{R}_w$. A subvariety $Y \subset X$ is an *integral variety* of \mathcal{R}_w if the smooth locus $Y^0 \subset Y$ is an integral manifold of \mathcal{R}_w . The Schur system is *rigid* if for every integral manifold M , there exists $g \in G$ such that $M \subset g \cdot X_w$. If every integral variety Y is of the form $g \cdot X_w$, then we say X_w is *Schur rigid*.

Remark. Similar to \mathcal{B}_w , we have that \mathcal{R}_w rigid implies X_w is Schur rigid.

The following theorem is proved in [17] and [3, §2.8.1].

Theorem 7.1. *A subvariety $Y \subset X$ is an integral variety of \mathcal{R}_w if and only if $[Y] = r[X_w] \in H_k(X)$ with $0 < r \in \mathbb{Z}$.*

The significance of the theorem in relation to our motivating question is that it implies: *if X_w is singular and Schur rigid, then X_w is not homologous to a smooth variety.*

Lemma 2.8(c) and Theorem 7.1 imply

Corollary 7.2. *$[X_w^*]$ is Poincaré dual to $[X_w]$.*

Proposition 7.3. *Let G/P be an irreducible compact Hermitian symmetric space. Let φ be an automorphism of the Dynkin diagram $\delta_{\mathfrak{g}}$ and $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the induced Lie algebra automorphism. Set $\mathfrak{p}' = \varphi(\mathfrak{p})$. Given $w \in W^{\mathfrak{p}}$, set $w' = \varphi(w) \in W^{\mathfrak{p}'}$. Then the Schubert system \mathcal{B}_w (resp. the Schur system \mathcal{R}_w) on G/P is rigid if and only if the Schubert system $\mathcal{B}_{w'}$ (resp. the Schur system $\mathcal{R}_{w'}$) on G/P' is rigid.*

Proof. The algebra isomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{g}$ induces a Lie group isomorphism $\varphi : G \rightarrow G$ with the property that $\varphi(P) = P'$. Thus, φ induces a biholomorphism $\varphi : G/P \rightarrow G/P'$. Let $\varphi_* : \text{Gr}(|w|, T(G/P)) \rightarrow \text{Gr}(|w'|, T(G/P'))$ denote the map induced by the push-forward. Review the definitions of *rigid* for the Schubert (Section 4.1) and Schur (above) systems.

Let $\mathfrak{g} = \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ denote the graded decomposition of \mathfrak{g} induced by \mathfrak{p}' (Section 2.2). Note that φ maps \mathfrak{g}_0 to \mathfrak{g}'_0 , inducing a Lie algebra isomorphism, and \mathfrak{g}_{-1} to \mathfrak{g}'_{-1} ; moreover, the latter is a $\mathfrak{g}_0 \simeq \mathfrak{g}'_0$ -module map. Additionally, $\Delta(w') = \varphi(\Delta(w))$, so that $\varphi(\mathbf{n}_w) = \mathbf{n}_{w'}$. Thus, $\varphi_*(\mathbf{I}_w) = \mathbf{I}_{w'}$. It follows that $\varphi_*(\mathcal{R}_w) = \mathcal{R}_{w'}$ and $\varphi_*(\mathcal{B}_w) = \mathcal{B}_{w'}$. \square

Remark 7.4. It may happen that $\bigwedge^{|w|} \mathfrak{g}_{-1}$ is irreducible as an \mathfrak{g}_0 -module, so that $R_w = \text{Gr}(|w|, \mathfrak{g}_{-1})$. In this case $H_{2|w|}(X) = \mathbb{Z}$ is generated by a single $[X_w]$. Every subvariety $Y \subset X$ of dimension $|w|$ is an integral variety of \mathcal{R}_w , so X_w cannot be Schur rigid. This is the case for (i) all w of length one; (ii) all w associated the CHSS $\mathbb{P}^n = A_n/P_1 \simeq A_n/P_n$ and $Q^{2n-1} = B_n/P_1$; and (iii) for all w such that $|w| \neq n-1$ associated to the CHSS $Q^{2n-2} = B_n/P_1$. By Theorem 6.1, none of these X_w satisfy H_+ .

Note that $B_w \subset R_w$ is the G_0 -orbit of the highest weight line $\mathbf{n}_w \in \mathbf{I}_w$. The following proposition was proven by Bryant in the case that X is a Grassmannian [3], and by Hong in the general setting [10, Proposition 2.10]. (The hypotheses of Hong's Proposition 2.10 are slightly more restrictive, but her proof establishes our Proposition 7.5.)

Proposition 7.5. *The Schur system \mathcal{R}_w is rigid if and only if the Schubert system \mathcal{B}_w is rigid and $B_w = R_w$.*

Having studied the Schubert system (Section 6), we now wish to determine when $B_w = R_w$.

7.2. Lemmas. We claim that R_w is connected. To see this note that each connected component of R_w contains a closed orbit in $\mathbb{P}\mathbf{I}_w$. Because the reductive \mathfrak{g}_0 acts irreducibly on \mathbf{I}_w , the space $\mathbb{P}\mathbf{I}_w$ contains a unique closed orbit. Hence R_w is connected. From this we deduce the following.

Lemma 7.6 ([3, 10]). *The equality $B_w = R_w$ holds if and only if $\widehat{T}_{\mathbf{n}_w} B_w = \widehat{T}_{\mathbf{n}_w} R_w$.*

Regard $\mathfrak{g}_{0,-}$ as a $\mathfrak{g}_{0,0}$ -submodule of $\mathbf{n}_w^\perp \otimes \mathbf{n}_w^*$. Since $\mathfrak{g}_{0,0}$ is reductive, there exists a $\mathfrak{g}_{0,0}$ -module \mathbf{t}_w with the property that $\mathbf{n}_w^\perp \otimes \mathbf{n}_w^* = \mathfrak{g}_{0,-} \oplus \mathbf{t}_w$. From the definition of ∂ it follows that

$$H_0^1(\mathbf{n}_w, \mathfrak{g}_w^\perp) \stackrel{(4.25)}{=} \frac{\mathbf{n}_w^\perp \otimes \mathbf{n}_w^*}{\text{im}\{\partial : \mathfrak{g}_{0,-} \rightarrow \mathbf{n}_w^\perp \otimes \mathbf{n}_w^*\}} = \mathbf{t}_w.$$

The identifications $\mathfrak{g}_{0,-} \simeq T_{\mathbf{n}_w} B_w$ and $\mathbf{n}_w^\perp \otimes \mathbf{n}_w^* \simeq T_{\mathbf{n}_w} \text{Gr}(|w|, \mathfrak{g}_{-1})$ yield $T_{\mathbf{n}_w} \text{Gr}(|w|, \mathfrak{g}_{-1}) \simeq T_{\mathbf{n}_w} B_w \oplus \mathbf{t}_w$. This establishes the following.

Lemma 7.7 ([10, Proposition 3.5]). *The graded cohomology $H_0^1(\mathbf{n}_w, \mathfrak{g}_w^\perp)$ is the $\mathfrak{g}_{0,0}$ -module complement of $T_{\mathbf{n}_w} B_w$ in $T_{\mathbf{n}_w} \text{Gr}(|w|, \mathfrak{g}_{-1})$.*

Let $\{\gamma_1, \gamma_2, \dots, \gamma_{|w|}\}$ be an enumeration of the roots $\Delta(w)$. Define

$$\langle w \rangle := \sum_j \gamma_j \quad \text{and} \quad \mathbf{v}_w := E_{-\gamma_1} \wedge \cdots \wedge E_{-\gamma_{|w|}} \in \mathbf{I}_w.$$

Then \mathbf{v}_w spans the highest weight line $\mathbf{n}_w \hookrightarrow \mathbb{P}\mathbf{I}_w$ of weight $-\langle w \rangle$. Note that $\xi \in \mathbf{n}_w^\perp \otimes \mathbf{n}_w^*$ naturally acts on \mathbf{v}_w by

$$\xi \mathbf{v}_w := E_{-\beta} \wedge (E_{\gamma} \lrcorner \mathbf{v}_w), \quad \text{when } \xi = E_{-\beta} \otimes E_{\gamma}.$$

Moreover,

$$\begin{aligned}\widehat{T}_{\mathbf{n}_w} \text{Gr}(|w|, \mathfrak{g}_{-1}) &= \text{span}\{\mathbf{v}_w, (\mathbf{n}_w^\perp \otimes \mathbf{n}_w^*)\mathbf{v}_w\} \subset \wedge^{|w|} \mathfrak{g}_{-1}, \\ \widehat{T}_{\mathbf{n}_w} R_w &= \mathbf{I}_w \cap \text{span}\{\mathbf{v}_w, (\mathbf{n}_w^\perp \otimes \mathbf{n}_w^*)\mathbf{v}_w\}, \\ \widehat{T}_{\mathbf{n}_w} B_w &= \text{span}\{\mathbf{v}_w, (\mathfrak{g}_{0,-})\mathbf{v}_w\}.\end{aligned}$$

Thus, $\widehat{T}_{\mathbf{n}_w} B_w = \widehat{T}_{\mathbf{n}_w} R_w$ (and $B_w = R_w$ by Lemma 7.6) if and only if $\mathbf{I}_w \cap (\mathfrak{t}_w)\mathbf{v}_w = \{0\}$.

Lemma 7.8 ([10, Proposition 3.4]). *The equality $B_w = R_w$ fails if and only if there exists a $\mathfrak{g}_{0,0}$ -highest weight vector $0 \neq \xi \in \mathfrak{t}_w = H_0^1(\mathbf{n}_w, \mathfrak{g}_w^\perp) \simeq \mathcal{H}_0^1$ with the property that $\xi\mathbf{v}_w \in \mathbf{I}_w$.*

Proof. Note that both \mathbf{I}_w and $(\mathfrak{t}_w)\mathbf{v}_w$ are $\mathfrak{g}_{0,0}$ -modules. Let $U \subset \mathfrak{t}_w$ be an irreducible $\mathfrak{g}_{0,0}$ -submodule with highest weight vector $\xi \in U$. Then either $U\mathbf{v}_w \subset \mathbf{I}_w$ or $\mathbf{I}_w \cap U\mathbf{v}_w = \{0\}$. The former holds if and only if $\xi\mathbf{v}_w \in \mathbf{I}_w$. \square

In Section 8 we apply Proposition 7.5 and Lemma 7.8 to show that the Schubert varieties satisfying \mathbf{H}_+ (Section 6) are Schur rigid.

Definition. Let $\Pi(w)$ be the set of pairs $(\gamma, \beta) \in \Delta(\mathfrak{g}_1) \times \Delta(\mathfrak{g}_1)$ such that:

- (1) the root γ is the highest weight of an irreducible $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{1,C} \subset \mathbf{n}_w^+$;
- (2) the root $-\beta$ is the highest weight of an irreducible $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{-1,-B} \subset \mathbf{n}_w^\perp$ (equivalently, β is a lowest weight of $\mathfrak{g}_{1,B}$); and
- (3) $\gamma - \beta$ is *not* a root of \mathfrak{g} .

Let $U_{\gamma-\beta} \subset \mathfrak{g}_{-1,-B} \otimes \mathfrak{g}_{1,C}$ denote the Cartan product.

Lemma 7.9. $\mathcal{H}_0^1 = \bigoplus_{(\gamma,\beta) \in \Pi(w)} U_{\gamma-\beta}$.

The proof of Lemma 7.9 is given in Section 7.3.

Given an element π of the $\mathfrak{g}_{0,0}$ -root lattice, let $\mathbf{I}_w^{-\pi} \subset \mathbf{I}_w$ be the weight space of weight $-\langle w \rangle - \pi$. By Lemma 7.8,

$$(7.10) \quad \begin{aligned} &\text{the equality } B_w = R_w \text{ fails if and only if} \\ &\text{there exists } (\gamma, \beta) \in \Pi(w) \text{ such that } \xi\mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}. \end{aligned}$$

Let $\mathbf{e} = \{\varepsilon_j\}_{j=1}^t \subset \Delta(\mathfrak{g}_0)$ be an ordered sequence. Define

$$\mathbf{e}.\mathbf{v}_w := E_{\varepsilon_t}(\dots E_{\varepsilon_2}(E_{\varepsilon_1}.\mathbf{v}_w)\dots) \in \mathbf{I}_w.$$

Lemma 7.11. *Fix $(\gamma, \beta) \in \Pi(w)$ and set $\mathbf{s} := (\beta - \gamma)(Z_w)$. The weight space $\mathbf{I}_w^{\gamma-\beta}$ is spanned by vectors of the form $\mathbf{b}.\mathbf{v}_w$ where \mathbf{b} ranges over ordered sequences $\mathbf{b} = \{-\beta_j\}_{j=1}^{\mathbf{s}} \subset \Delta(\mathfrak{g}_{0,-1})$ such that $\beta - \gamma = \sum_1^{\mathbf{s}} \beta_j$. In particular, $\beta - \gamma$ is a sum of positive roots, and $\mathbf{s} > 1$.*

Remark 7.12. Suppose $\mathbf{a}(w) = 0$ and $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{-1,-1}$. The condition $\mathbf{s} > 1$ of Lemma 7.11 implies that $B_w = R_w$ holds.

Proof of Lemma 7.11. By standard weight theory, $\mathbf{I}_w^{-\pi}$ is spanned by vectors of the form $\mathbf{e}.\mathbf{v}_w$ where \mathbf{e} ranges over all ordered sequences $\{-\varepsilon_j\}_{j=1}^t$, with each ε_j a simple root and $-\langle w \rangle - \sum_j \varepsilon_j = -\langle w \rangle - \pi$; equivalently $\sum_j \varepsilon_j = \pi$. (Here t may vary with the choice of sequence $\{-\varepsilon_j\}$.)

Note that ε_j must lie in $\Delta(\mathfrak{g}_{0,1}) \sqcup \Delta(\mathfrak{g}_{0,0})$. Since $\mathfrak{g}_{0,\geq 0}$ is the stabilizer of \mathbf{n}_w in \mathfrak{g}_0 , we have $E_{-\varepsilon_j} \cdot \mathbf{v}_w = 0$ if and only if $\varepsilon_j \in \Delta(\mathfrak{g}_{0,0})$. So, without loss of generality, $\varepsilon_1 \in \Delta(\mathfrak{g}_{0,1})$. If $\varepsilon_2 \in \Delta(\mathfrak{g}_{0,0})$, then

$$\begin{aligned} E_{-\varepsilon_2} \cdot (E_{-\varepsilon_1} \cdot \mathbf{v}_w) &= [E_{-\varepsilon_2}, E_{-\varepsilon_1}] \cdot \mathbf{v}_w + E_{-\varepsilon_1} \cdot (E_{-\varepsilon_2} \cdot \mathbf{v}_w) \\ &= [E_{-\varepsilon_2}, E_{-\varepsilon_1}] \cdot \mathbf{v}_w + c E_{-\varepsilon_1} \cdot \mathbf{v}_w, \end{aligned}$$

for some $c \in \mathbb{C}$. Either $[E_{-\varepsilon_2}, E_{-\varepsilon_1}] = 0$, or $\varepsilon_1 + \varepsilon_2 \in \Delta(\mathfrak{g}_{0,1})$. It follows now from an inductive argument that the vectors $\mathbf{b} \cdot \mathbf{v}_w$ span $\mathbf{I}_w^{-\pi}$.

Fix $(\gamma, \beta) \in \Pi(w)$ with $\beta - \gamma = \pi$. The inequality $\mathbf{s} \geq 1$ follows from the definition of $\Pi(w)$. If equality holds, $\beta - \gamma = \beta_1$ is a root. This contradicts the definition of $\Pi(w)$. \square

Remark 7.13. Suppose that (a) for every ordered sequence \mathbf{b} of Lemma 7.11 it is the case that $[E_{\beta_j}, E_{\beta_k}] = 0$ for all $1 \leq j, k \leq \mathbf{s}$, and (b) for any every pair \mathbf{b} and \mathbf{b}' of ordered sequences, the unordered sets are equal. Then $\mathbf{b} \cdot \mathbf{v}_w = \mathbf{b}' \cdot \mathbf{v}_w$. It follows that $\dim \mathbf{I}_w^{-\pi} = 1$.

7.3. Proof of Lemma 7.9. Given $0 < \ell \in \mathbb{Z}$, let $H_{0,-\ell}^1$ be the component of $H_0^1(\mathbf{n}_w, \mathfrak{g}_w^\perp)$ of Z_w -degree $-\ell$. Then

$$(7.14) \quad H_0^1(\mathbf{n}_w, \mathfrak{g}_w^\perp) = \bigoplus_{0 < \ell} H_{0,-\ell}^1.$$

Fix $0 < \ell \in \mathbb{Z}$, let T^k be the component of $\mathfrak{g}_w^\perp \otimes \wedge^k \mathbf{n}_w^+$ of (Z_i, Z_w) -bidegree $(0, -\ell)$. Then

$$(7.15) \quad \begin{aligned} T^0 &= \mathfrak{g}_{0,-\ell}, \\ T^1 &= T_0^1 \oplus T_1^1 \oplus \cdots \oplus T_{\ell-1}^1, \quad \text{with } T_m^1 := \mathfrak{g}_{-1,m-\mathbf{a}-\ell} \otimes \mathfrak{g}_{1,\mathbf{a}-m}, \end{aligned}$$

and $T^k = \{0\}$ for all $k \neq 0, 1$. Since the Lie algebra cohomology differential preserves the (Z_i, Z_w) -bidegree, we have a subcomplex

$$(7.16) \quad \{0\} \longrightarrow T^0 \xrightarrow{\partial} T^1 \longrightarrow \{0\},$$

and

$$H_{0,-\ell}^1 = T^1 \bmod \partial T^0 =: H^1(T^\bullet, \partial).$$

Let ∂_m denote the component of $\partial : T^0 \rightarrow T^1$ taking value in T_m^1 . Consider the associated subcomplex

$$\{0\} \longrightarrow T^0 \xrightarrow{\partial_m} T_m^1 \longrightarrow \{0\}.$$

Note that ∂_m is a $\mathfrak{g}_{0,0}$ -module map. Therefore, the cohomology

$$(7.17) \quad H^1(T^\bullet, \partial_m) := T_m^1 \bmod \partial_m(T^0)$$

is a $\mathfrak{g}_{0,0}$ -module.

Lemma 7.18. *The $\mathfrak{g}_{0,0}$ -modules $H_{0,-\ell}^1 = T^1 / \partial(T^0)$ and $\oplus_m H^1(T^\bullet, \partial_m)$ are isomorphic.*

The lemma is proved in Section 7.4. We continue here with a discussion of the consequences. Observe that $\partial_m^* := \mathbf{d}_{|T_m^1}^* : T_m^1 \rightarrow T^0$ is adjoint to ∂_m with respect to the positive definite Hermitian inner product $\langle \cdot, \cdot \rangle$ of Section 5.1. Let

$$\square_m := \partial_m \circ \partial_m^* : T_m^1 \rightarrow T_m^1$$

denote the associated Laplacian. Analogous to Proposition 5.10, there exists a $\mathfrak{g}_{0,0}$ -module isomorphism $H^1(T^\bullet, \partial_m) \simeq \mathcal{H}_{\partial_m}^1 := \ker \square_m$. Therefore, corollary to Lemma 7.18, we have

$$(7.19) \quad H_{0,-\ell}^1 \simeq \bigoplus_{m=0}^{\ell-1} \mathcal{H}_{\partial_m}^1$$

as $\mathfrak{g}_{0,0}$ -modules

Section 5.2 yields the following expression

$$\begin{aligned} \square_m &\stackrel{(5.16)}{=} \sum_{\alpha \in \Delta(w, \mathfrak{a}-m)} \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial_m^* \\ &\stackrel{(3.2)}{=} \sum_{\alpha \in \Delta(w, \mathfrak{a}-m)} \left(\epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \circ \partial_m^* + \partial^* \circ \epsilon_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} \right) \\ &\stackrel{(*)}{=} \sum_{\alpha \in \Delta(w, \mathfrak{a}-m)} \mathcal{L}'_{E_\alpha} \circ \mathcal{L}'_{E_{-\alpha}} - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j} \stackrel{(3.2)}{=} - \sum_{\zeta_j \in \mathfrak{g}_{0,0}} \mathcal{L}''_{\zeta_j} \circ \mathcal{L}'_{\xi_j}. \end{aligned}$$

The equality $(*)$ is obtained from an application of Lemma 5.15(b), followed by applications of Lemma 5.15(a,c).

Let $\mathfrak{g}_{-1,-B} \subset \mathfrak{g}_{-1,m-\mathfrak{a}-\ell}$ and $\mathfrak{g}_{1,C} \subset \mathfrak{g}_{1,\mathfrak{a}-m}$ be irreducible $\mathfrak{g}_{0,0}$ -modules of highest weights $-\beta \in \Delta(\mathfrak{g}_{-1,-B})$ and $\gamma \in \Delta(\mathfrak{g}_{1,C})$, respectively. Let $U \subset \mathfrak{g}_{-1,-B} \otimes \mathfrak{g}_{1,C} \subset T_m^1$ be an irreducible module of highest weight π . By Lemma 5.23, the Laplacian \square_m acts on U by a scalar $c \geq (\beta, \gamma)$, with equality if and only if $\pi = \gamma - \beta$. By (3.2) $\gamma + \beta \notin \Delta(\mathfrak{g})$. So $(\gamma, \beta) \geq 0$, and equality holds if and only if $\gamma - \beta \notin \Delta(\mathfrak{g})$. Thus $U = U_\pi \subset \mathcal{H}_{\partial_m}^1$ if and only if $\pi = \gamma - \beta$ for some $(\gamma, \beta) \in \Pi(w)$. Given (7.19), this establishes Lemma 7.9.

7.4. Proof of Lemma 7.18. The proof of the lemma is by induction. Define a filtration $\{0\} \subset F^1 T^\bullet \subset F^0 T^\bullet = T^\bullet$ of the complex (7.16) by

$$F^1 T^0 = \{0\}, \quad F^1 T^1 = T_1^1 \oplus \cdots \oplus T_{\ell-1}^1.$$

We may identify $\text{Gr}^0 T^1 := F^0 T^1 / F^1 T^1$ with T_0^1 as $\mathfrak{g}_{0,0}$ -modules. Consider the associated spectral sequence $\{E_r^{p,q}\}$. (See [7, Chapter 3.5] for a treatment of the spectral sequence associated to a filtered complex.) We have $E_2^{p,q} = E_\infty^{p,q}$ with $E_\infty^{0,0} = \ker\{\partial : T^0 \rightarrow T^1\}$,

$$E_\infty^{0,1} = \frac{T^1}{F^1 T^1 + \partial T^0}, \quad \text{and} \quad E_\infty^{1,0} = \frac{F^1 T^1}{F^1 T^1 \cap \partial T^0}$$

and all other $E_\infty^{p,q} = \{0\}$. The vector space identification (see [7, Chapter 3.5])

$$H_{0,-\ell}^1 = H^1(T^\bullet, \partial) \simeq \text{Gr}^0 H^1(T^\bullet, \partial) \oplus \text{Gr}^1 H^1(T^\bullet, \partial) = E_\infty^{0,1} \oplus E_\infty^{1,0}$$

is a $\mathfrak{g}_{0,0}$ -module identification. Additionally, $E_\infty^{0,1} \simeq H^1(T^\bullet, \partial_0)$ as $\mathfrak{g}_{0,0}$ -modules, cf. (7.17).

The module $E_\infty^{1,0}$ is described as follows. Let $F^1 \partial = \partial_1 + \cdots + \partial_{\ell-1}$ denote the component of $\partial : T^0 \rightarrow T^1$ taking values in $F^1 T^1$, with respect to the decomposition (7.15). Then

$$\{0\} \longrightarrow \mathcal{C}^0 := T^0 \xrightarrow{F^1 \partial} \mathcal{C}^1 := F^1 T^1 \longrightarrow \{0\}$$

is a complex, and $E_\infty^{1,0} = H^1(\mathcal{C}^\bullet, F^1 \partial)$. Thus,

$$(7.20) \quad H_{0,-\ell}^1 \simeq H^1(T^\bullet, \partial_0) \oplus H^1(\mathcal{C}^\bullet, F^1 \partial).$$

This completes the first step in the induction. In order to state the inductive hypothesis let $0 \leq m < \ell$ and define $\mathcal{C}_m^0 := T^0$ and

$$(7.21) \quad \mathcal{C}_m^1 := T_m^1 \oplus \underbrace{T_{m+1}^1 \oplus \cdots \oplus T_{\ell-1}^1}_{=: F^1 \mathcal{C}_m^1}.$$

Let $F^m \partial = \partial_m + \cdots + \partial_{\ell-1}$ denote the component of $\partial : T^0 \rightarrow T^1$ taking values in \mathcal{C}_m^1 , with respect to the decomposition (7.15). Then

$$(7.22) \quad 0 \longrightarrow T^0 \xrightarrow{F^m \partial} \mathcal{C}_m^1 \longrightarrow \{0\}$$

is a complex; let $H^1(\mathcal{C}_m^\bullet, F^m \partial)$ denote the associated cohomology. Observe that

$$(7.23) \quad H^1(T^\bullet, \partial) = \left(\bigoplus_{m < r} H^1(T^\bullet, \partial_m) \right) \oplus H^1(\mathcal{C}_r^\bullet, F^r \partial).$$

holds for $r = 1$ by (7.20). Inductive hypothesis: (7.23) holds for some $r = r_o$ with $0 < r_o < \ell - 1$. We will show that (7.23) holds for $r = r_o + 1$. The lemma will then follow.

Define a filtration $\{0\} \subset F^1 \mathcal{C}_m^\bullet \subset F^0 \mathcal{C}_m^\bullet := \mathcal{C}_m^\bullet$ on the complex (7.22) by $F^1 \mathcal{C}_m^0 = \{0\}$ and (7.21). Spectral sequence computations identical to those yielding (7.20) produce

$$H^1(\mathcal{C}_m^\bullet, F^m \partial) = H^1(T^\bullet, \partial_m) \oplus H^1(\mathcal{C}_{m+1}^\bullet, \partial^{m+1}).$$

Combined with the inductive hypothesis, this implies that (7.23) holds for $r = r_o + 1$. This completes the induction and establishes the lemma.

8. SCHUR RIGIDITY

Theorem 6.1 lists the proper Schubert varieties which satisfy H_+ ; these are the varieties for which there exist first-order obstructions to Schubert flexibility. The main result of the paper is

Theorem 8.1. *The varieties listed in Theorem 6.1 are Schur rigid.*

Outline of proof of Theorem 8.1. By hypothesis, in all sections to follow, X_w will be assumed to satisfy H_+ , hence is listed in Theorem 6.1 and is Schubert rigid. By Proposition 7.5, it suffices to show that the equality $B_w = R_w$ holds. This is done in the sections that follow. The computational tools are (7.10) and Lemma 7.11. \square

Be aware that the action of \mathbf{b} on $\bigwedge^{|w|} \mathfrak{g}_{-1}$ is induced by the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_{-1} , while the action of $\xi = E_{-\beta} \otimes E_\gamma \in \mathfrak{n}_w^\perp \otimes \mathfrak{n}_w^*$ is given by $\epsilon_{E_{-\beta}} \circ \iota_{E_\gamma}$; that is $\xi \mathbf{v}_w := E_{-\beta} \wedge (E_\gamma \lrcorner \mathbf{v}_w)$.

8.1. Comparison with Hong's results. Before proving the theorem, we discuss the relationship of Theorem 8.1 to the main results in [10, 9]. In the smooth case ($\mathbf{a} = 0$) we recover precisely Hong's result in Section 1.2; see Table 7.

The main result of [9] is

Theorem (Hong). *Let $\pi = (p_1^{q_1}, \dots, p_r^{q_r}) \in \mathbf{P}(\mathbf{i}, n+1)$, $p_r \neq 0$ with conjugate $\pi' = (p_1^{q'_1}, \dots, p_r^{q'_r}) \in \mathbf{P}(n+1-\mathbf{i}, n+1)$, $p'_r \neq 0$. If $q_i, q'_i \geq 2$ for all $1 \leq i \leq r$, then the Schubert variety $X_\pi \subset \mathrm{Gr}(\mathbf{i}, n+1) = A_n/P_{\mathbf{i}}$ is Schur rigid.*

TABLE 7. The proper smooth Schubert varieties satisfying H_+ .

G/P	J	Description of X_w
A_n/P_i	$\{i-1\} \& \{i+1\}$	maximal linear subspaces \mathbb{P}^{n-i+1} & \mathbb{P}^i , resp.
	$\{j_1, j_2\}$	$\text{Gr}(i-j_1, j_2-j_1)$
	with $1 < i < n$ and $1 < i-j_1, j_2-i$.	
D_n/P_1	$\{n-1\} \& \{n\}$	maximal linear subspaces \mathbb{P}^{n-1}
C_n/P_n	$\{j\}$	C_{n-j}/P_{n-j} with $j < n-1$
D_n/P_n	$\{j\}$	D_{n-j}/P_{n-j} , if $j < n-3$;
		\mathbb{P}^3 with $j = n-3$, and \mathbb{P}^{n-1} with $j = n-1$
E_6/P_6	$\{1\}, \{2\} \& \{3\}$	$Q^8 = D_5/P_1, \mathbb{P}^5$ & \mathbb{P}^4 , resp.
E_7/P_7	$\{1\}, \{2\} \& \{3\}$	$Q^{10} = D_6/P_1, \mathbb{P}^6$ & \mathbb{P}^5 , resp.

Hong's theorem assumes, in particular, that $1 < q_1, q'_1$. Comparing with Remark 6.3 we see that Hong's theorem omits some Schur rigid X_π with $\pi \in \spadesuit, \heartsuit$ and \diamondsuit .

8.2. Grassmannian $\text{Gr}(i, n+1) = A_n/P_i$. In this section we will prove that $B_w = R_w$ holds for the varieties in Theorem 6.1(a). By Proposition 7.5 it suffices to show that $R_w = B_w$ when condition H_+ holds.

The Schubert variety X_w is given by Corollary 3.17. In the case that $|J| = 1$, we have $\mathfrak{g}_{-1} = \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{-1,-1}$, and $R_w = B_w$ is given by Remark 7.12.

Next consider the case that $|J| > 1$. By the lemma below we may assume $(\beta - \gamma)(Z_w) = 2$.

Lemma 8.2. *The equality $B_w = R_w$ fails if and only if there exists $(\gamma, \beta) \in \Pi(w)$ with $(\beta - \gamma)(Z_w) = 2$ such that $\xi \mathbf{v}_w = E_{-\beta} \wedge (E_{\gamma \lrcorner} \mathbf{v}_w) \in \mathbf{I}_w^{\gamma-\beta}$.*

The lemma is proven below. It also holds for C_n and D_n , but not E_6 and E_7 .

Recall that $\beta - \gamma$ is a sum of positive roots (Lemma 7.11). Since β is a lowest $\mathfrak{g}_{0,0}$ -weight, it is of the form $\beta = \alpha_j + \cdots + \alpha_{j'}$, where $j, j' \in J$ and $j < i < j'$. Similarly, since γ is a highest $\mathfrak{g}_{0,0}$ -weight, it is of the form $\gamma = \alpha_{k+1} + \cdots + \alpha_{k'-1}$, where $k, k' \in J \cup \{0, n+1\}$ and $k < i < k'$. The condition that $\beta - \gamma$ be a sum of positive roots, but not a root itself, implies that $j \leq k$ and $k' \leq j'$. Moreover, $(\beta - \gamma)(Z_w) = 2$ forces $j = k$ and $k' = j'$ so that $\beta - \gamma = \alpha_j + \alpha_{j'}$. By Remark 7.13 the weight space $\mathbf{I}_w^{\gamma-\beta}$ is one-dimensional and spanned by $\mathbf{b} \cdot \mathbf{v}_w$, where $\mathbf{b} = \{-\alpha_j, -\alpha_{j'}\}$. From Lemma 8.2 we conclude that

$$(8.3) \quad B_w = R_w \text{ holds if and only if } \mathbf{b} \cdot \mathbf{v}_w \text{ and } \xi \cdot \mathbf{v}_w \text{ are linearly independent for all } (\gamma, \beta) \in \Pi(w) \text{ with } (\beta - \gamma)(Z_w) = 2.$$

Define $\beta_1, \beta_2 \in \Delta(\mathfrak{g}_{0,1})$ by $\mathbf{b} = \{-\beta_1, -\beta_2\}$. A priori

$$E_{-\beta_1} \cdot \mathbf{v}_w = \sum_{\substack{\nu \in \Delta(w) \\ \nu + \beta_1 \in \Delta}} c_\nu E_{-\nu - \beta_1} \wedge (E_{\nu \lrcorner} \mathbf{v}_w),$$

for some $c_\nu \in \mathbb{C}$. However, since $E_{-\nu-\beta_1} \in \mathfrak{n}_w$ for all ν such that $\nu(Z_w) < \mathfrak{a}$, we have

$$E_{-\beta_1} \cdot \mathbf{v}_w = \sum_{\substack{\nu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \\ \nu + \beta_1 \in \Delta}} c_\nu E_{-\nu-\beta_1} \wedge (E_\nu \lrcorner \mathbf{v}_w) = \sum_{\substack{\nu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \\ \nu + \beta_1 \in \Delta}} c_\nu (E_{-\nu-\beta_1} \otimes E_\nu) \cdot \mathbf{v}_w$$

for nonzero coefficients c_ν . Similarly,

$$(8.4) \quad \begin{aligned} \mathbf{b} \cdot \mathbf{v}_w &= \sum_{\substack{\nu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \\ \nu + \beta_1 \in \Delta \\ \nu + \beta_1 + \beta_2 \in \Delta}} c_\nu^1 E_{-\nu-\beta_1-\beta_2} \wedge (E_\nu \lrcorner \mathbf{v}_w) + \sum_{\substack{\nu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \\ \nu + \beta_1 \in \Delta \\ \mu = \nu - \beta_2 \in \Delta}} c_\nu^2 E_{-\nu-\beta_1} \wedge (E_\mu \lrcorner \mathbf{v}_w) \\ &+ \sum_{\substack{\nu, \mu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \\ \mu + \beta_2, \nu + \beta_1 \in \Delta}} c_{\nu, \mu} (E_{-\nu-\beta_1} \wedge E_{-\beta_2-\mu}) \wedge ((E_\nu \wedge E_\mu) \lrcorner \mathbf{v}_w), \end{aligned}$$

for nonzero coefficients c_ν^1 , c_ν^2 and $c_{\nu, \mu}$.

Consider the first sum of (8.4). The condition that both $\nu + \beta_1$ and $\nu + \beta_1 + \beta_2$ be roots uniquely determines ν : it must be the case that $\nu = \gamma$. Thus, the sum is empty if $\gamma(Z_w) = \mathfrak{a} - 1$, and a nonzero multiple of $\xi \mathbf{v}_w$ if $\gamma(Z_w) = \mathfrak{a}$.

Next observe that second sum of (8.4) is over the set of all

$$\{\mu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}-1}) \mid \mu + \beta_2, \mu + \beta_1 + \beta_2 \in \Delta\}.$$

Again, the condition that both $\mu + \beta_2$ and $\mu + \beta_2 + \beta_1$ be roots uniquely determines μ : it must be the case that $\mu = \gamma$. So the sum is a nonzero multiple of $\xi \mathbf{v}_w$ if $\gamma(Z_w) = \mathfrak{a} - 1$, and empty otherwise.

From the observations above we conclude that $\mathbf{b} \cdot \mathbf{v}_w$ and $\xi \mathbf{v}_w$ are linearly independent if and only if the third sum of (8.4) is nonzero. Equivalently,

$$(8.5) \quad \begin{aligned} &\mathbf{b} \cdot \mathbf{v}_w \text{ and } \xi \mathbf{v}_w \text{ are linearly independent if and only if there exist} \\ &\text{distinct } \nu, \mu \in \Delta(\mathfrak{g}_{1,\mathfrak{a}}) \text{ such that } \nu + \beta_1 \text{ and } \mu + \beta_2 \text{ are distinct roots.} \end{aligned}$$

There are two cases to consider: $\beta(Z_w) = \mathfrak{a} + 2$ and $\gamma(Z_w) = \mathfrak{a}$; or $\beta(Z_w) = \mathfrak{a} + 1$ and $\gamma(Z_w) = \mathfrak{a} - 1$. Suppose that $\mathbf{j} = \mathbf{j}_r$ and $\mathbf{j}' = \mathbf{j}_s$. In the first case, the μ, ν of (8.5) exist if and only if one of the following holds:

$$\begin{aligned} &1 < \mathbf{j}_{r+1} - \mathbf{j}_r \quad (\text{or } \mathbf{i} - \mathbf{j}_r > 1 \text{ if } \mathbf{j}_r < \mathbf{i} < \mathbf{j}_{r+1}), \quad \text{or} \\ &1 < \mathbf{j}_s - \mathbf{j}_{s-1} \quad (\text{or } \mathbf{j}_s - \mathbf{i} > 1 \text{ if } \mathbf{j}_{s-1} < \mathbf{i} < \mathbf{j}_s). \end{aligned}$$

In the second case the μ, ν of (8.5) exist if and only if one of the following holds:

$$1 < \mathbf{j}_r - \mathbf{j}_{r-1} \quad \text{or} \quad 1 < \mathbf{j}_{s+1} - \mathbf{j}_s.$$

(If $r = 1$, then $\mathbf{j}_0 := 0$; if $s = |\mathbf{J}|$, then $\mathbf{j}_{s+1} := n$.)

Modulo the proof of Lemma 8.2, this completes the proof of the proposition.

Proof of Lemma 8.2 in the case that $G = A_n$. Given (7.10), it suffices to show the following: if there exists $(\beta, \gamma) \in \Pi(w)$ such that $\xi \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}$, then there exists $(\beta_o, \gamma_o) \in \Pi(w)$ such that $\xi_o \mathbf{v}_w \in \mathbf{I}_w^{\gamma_o-\beta_o}$ and $(\beta_o - \gamma_o)(Z_w) = 2$.

The condition $(\beta - \gamma)(Z_w) > 2$ holds if and only if either $\mathbf{j} < \mathbf{k}$ or $\mathbf{k}' < \mathbf{j}'$. Suppose that $\mathbf{j} = \mathbf{j}_r$ and $\mathbf{j} < \mathbf{k}$. Suppose that $\beta(Z_w) > \mathfrak{a} + 1$. Set $\mathbf{e} = \{\alpha_{\mathbf{j}_r}, \alpha_{\mathbf{j}_r+1}, \dots, \alpha_{\mathbf{j}_{r+1}-1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $E_{-\beta_o} \wedge (E_\gamma \lrcorner \mathbf{v}_w) = \xi_o \mathbf{v}_w$ where $\beta_o = \alpha_{\mathbf{j}_{r+1}} + \dots + \alpha_{\mathbf{j}'}$ $\in \Delta(\mathfrak{g}_1)$

is, like β , a lowest $\mathfrak{g}_{0,0}$ -weight, and $\beta_o(Z_w) = \beta(Z_w) - 1$. In particular, if $\xi \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}$, then $\xi_o \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta_o}$. Continuing inductively, we may assume that either $j = k$, or $\beta(Z_w) = a + 1$.

Similarly, if $k' < j' = j_s$ and $\beta(Z_w) > a + 1$, set $\mathbf{e} = \{\alpha_{j_s}, \alpha_{j_s-1}, \dots, \alpha_{j_s-1+1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $E_{-\beta_o} \wedge (E_{\gamma \lrcorner} \mathbf{v}_w) = \xi_o \mathbf{v}_w$ where $\beta_o = \alpha_j + \dots + \alpha_{j_s-1} \in \Delta(\mathfrak{g}_1)$ is, like β , a lowest $\mathfrak{g}_{0,0}$ -weight, and $\beta_o(Z_w) = \beta(Z_w) - 1$. Again, if $\xi \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}$, then $\xi_o \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta_o}$. Continuing inductively, we may assume that either $j = k$ and $j' = k'$, or $\beta(Z_w) = a + 1$.

If $j = k$ and $j' = k'$, then $(\beta - \gamma)(Z_w) = 2$. So assume that $j < k$ and $\beta(Z_w) = a + 1$. Set $k = j_t$ and $\mathbf{e} = \{\alpha_{j_t}, \alpha_{j_t-1}, \dots, \alpha_{j_t-1+1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $\xi_o \mathbf{v}_w = E_{-\beta} \wedge (E_{\gamma_o \lrcorner} \mathbf{v}_w)$, where $\gamma_o = \alpha_{j_t-1+1} + \dots + \alpha_{k'}$ is, like γ , a highest $\mathfrak{g}_{0,0}$ -weight, and $\gamma_o(Z_w) = \gamma(Z_w) + 1$. As above, if $\xi \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}$, then $\xi_o \mathbf{v}_w \in \mathbf{I}_w^{\gamma_o-\beta}$. Continuing inductively we may assume that $j = k$.

By an analogous argument we may assume that $j' = k'$. \square

8.3. Quadric hypersurface $Q^{2n-2} = D_n/P_1$. It is an immediate consequence of Remark 7.12 that $B_w = R_w$ holds for the varieties in Theorem 6.1(b).

8.4. Lagrangian grassmannian C_n/P_n . In this section we show that $B_w = R_w$ holds for the varieties in Theorem 6.1(c).

Let $(\gamma, \beta) \in \Pi(w)$. As a lowest $\mathfrak{g}_{0,0}$ -weight β is of one of the following forms

$$(8.6a) \quad \beta = \alpha_{j_r} + \dots + \alpha_{j_s-1} + 2(\alpha_{j_s} + \dots + \alpha_{n-1}) + \alpha_n,$$

$$(8.6b) \quad \beta = 2(\alpha_{j_s} + \dots + \alpha_{n-1}) + \alpha_n, \quad \beta = \alpha_{j_r} + \dots + \alpha_n$$

for some $j_r, j_s \in J = \{j_1, \dots, j_p\}$. (Cf. the proof of Lemma 3.15. Also, Corollary 3.17 asserts that $a \leq p \leq a + 1$.) Similarly, as a $\mathfrak{g}_{0,0}$ -highest weight, γ is of one of the following forms

$$(8.7a) \quad \gamma = \alpha_{j_t+1} + \dots + \alpha_{j_u} + 2(\alpha_{j_u+1} + \dots + \alpha_{n-1}) + \alpha_n,$$

$$(8.7b) \quad \gamma = 2(\alpha_{j_u+1} + \dots + \alpha_{n-1}) + \alpha_n, \quad \gamma = \alpha_{j_t+1} + \dots + \alpha_n$$

for some $j_t, j_u \in J \cup \{0\}$, with $j_0 := 0$. (The third γ assumes that $n - 1 \in J$.) The requirement that $\beta - \gamma$ be a sum of positive roots, but not a root itself, rules out the second root in (8.6b).

Lemma 8.2 also holds for $G = C_n$. The proof is given at the end of this section. The condition $(\beta - \gamma)(Z_w) = 2$ implies that the pair (γ, β) is of one of the two following forms

$$\beta = \gamma + \alpha_j + \alpha'_j = \alpha_j + \dots + \alpha_{j'-1} + 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n,$$

$$\text{or } \beta = \gamma + 2\alpha_j = 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n.$$

We express $\beta - \gamma$ uniformly as $\alpha_j + \alpha'_j$ with $j \leq j' \in J$. The condition that $\beta - \gamma$ is not a root forces $0 \leq j' - j \neq 1$.

Set $\mathbf{b} = \{-\alpha_j, -\alpha_{j'}\} =: \{-\beta_1, -\beta_2\}$. By Remark 7.13 the weight space $\mathbf{I}_w^{\gamma-\beta}$ is one-dimensional and spanned by $\mathbf{b} \cdot \mathbf{v}_w$. In particular, (8.3) holds. The vector $\mathbf{b} \cdot \mathbf{v}_w$ is given by (8.4). Arguments analogous to those of Section 8.2 yield (8.5). Suppose $j = j_r$ and $j' = j_s$. There are two cases to consider:

(A) Suppose $\beta(Z_w) = \mathbf{a} + 2$ and $\gamma(Z_w) = \mathbf{a}$. First assume $1 < j' - j$. The pair μ, ν of (8.5) exists if and only if at least one of the following holds:

$$(8.8) \quad 1 < j_{r+1} - j_r \quad \text{or} \quad 1 < j_{s+1} - j_s. \quad (\text{If } s = |J|, \text{ then } j_{s+1} := n.)$$

Next suppose $j' = j$. The pair μ, ν exists if and only if (8.8) holds.

(B) Suppose $\beta(Z_w) = \mathbf{a} + 1$ and $\gamma(Z_w) = \mathbf{a} - 1$. First assume $1 < j' - j$. The pair μ, ν of (8.5) exists if and only if at least one of the following holds:

$$(8.9) \quad 1 < j_r - j_{r-1} \quad \text{or} \quad 1 < j_s - j_{s-1}. \quad (\text{If } r := 1, \text{ then } j_0 := 0.)$$

Next suppose $j' = j$. The pair μ, ν exists if and only if (8.9) holds.

Modulo Lemma 8.2, which is proven below, this establishes the proposition.

Proof of Lemma 8.2 for $G = C_n$. Given (7.10), it suffices to show the following: if there exists $(\beta, \gamma) \in \Pi(w)$ such that $\xi \mathbf{v}_w \in \mathbf{I}_w^{\gamma-\beta}$, then there exists $(\beta_o, \gamma_o) \in \Pi(w)$ with $(\beta_o - \gamma_o)(Z_w) = 2$ and $\xi_o \mathbf{v}_w \in \mathbf{I}_w^{\gamma_o-\beta_o}$.

(a.a) Begin with the case that β and γ are of the form (8.6a) and (8.7a), respectively. The condition that $\beta - \gamma$ be a sum of positive roots, but not a root itself, implies that either $j_q < j_s + 1 < j_r < j_t + 1$, or $j_r < j_s + 1$.

(a.a.1) Assume $j_q < j_s + 1 < j_r < j_t + 1$. Suppose that $\beta(Z_w) > \mathbf{a} + 1$ and $j_q < j_s$. Set $\mathbf{e} = \{\alpha_{j_q}, \alpha_{j_q+1}, \dots, \alpha_{j_{q+1}-1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $E_{-\beta_o} \wedge (E_{\gamma \lrcorner} \mathbf{v}_w) = \xi_o \mathbf{v}_w$, where $\beta_o = \alpha_{j_q+1} + \dots + \alpha_{j_r-1} + 2(\alpha_{j_r} + \dots + \alpha_{n-1}) + \alpha_n$. In particular, β_o is also a $\mathfrak{g}_{0,0}$ -lowest weight, and $\beta_o(Z_w) = \beta(Z_w) - 1$. Continuing inductively, we may assume that either $(\beta - \gamma)(Z_w) = 2$, or $\beta(Z_w) = \mathbf{a} + 1$, or $j_q = j_s$. In the first case we are done.

Suppose $\beta(Z_w) > \mathbf{a} + 1$ and $j_r < j_t$, and set $\mathbf{e} = \{\alpha_{j_r}, \alpha_{j_r+1}, \dots, \alpha_{j_{r+1}-1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $E_{-\beta_o} \wedge (E_{\gamma \lrcorner} \mathbf{v}_w) = \xi_o \mathbf{v}_w$, where $\beta_o = \alpha_{j_q} + \dots + \alpha_{j_{r+1}-1} + 2(\alpha_{j_{r+1}} + \dots + \alpha_{n-1}) + \alpha_n$. In particular, β_o is also a $\mathfrak{g}_{0,0}$ -lowest weight, and $\beta_o(Z_w) = \beta(Z_w) - 1$. Continuing inductively, we may assume that either $\beta(Z_w) = \mathbf{a} + 1$, or both $j_q = j_s$ and $j_r = j_t$. In the latter case $(\beta - \gamma)(Z_w) = 2$ and we are done. So we assume $\beta(Z_w) = \mathbf{a} + 1$.

If $j_q < j_s$, set $\mathbf{e} = \{\alpha_{j_s}, \alpha_{j_s-1}, \dots, \alpha_{j_{s-1}+1}\}$. Then $\mathbf{e} \cdot (\xi \mathbf{v}_w)$ is a nonzero multiple of $E_{-\beta} \wedge (E_{\gamma_o \lrcorner} \mathbf{v}_w) = \xi_o \mathbf{v}_w$ where $\gamma_o = \alpha_{j_{s-1}+1} + \dots + \alpha_{j_t} + 2(\alpha_{j_{t+1}} + \dots + \alpha_{n-1}) + \alpha_n$. Note that γ_o is, like γ , a $\mathfrak{g}_{0,0}$ -highest weight, and $\gamma_o(Z_w) = \gamma(Z_w) + 1$. Continuing inductively, we may assume that $j_q = j_s$.

It remains to consider the case that $j_r < j_t$. As above, an inductive argument, with $\mathbf{e} = \{\alpha_{j_t}, \alpha_{j_t-1}, \dots, \alpha_{j_{t-1}+1}\}$, reduces us to the case that $j_q = j_s$ and $j_r = j_t$. To summarize: we have produced an element $\xi_o \mathbf{v}_w$ in the \mathfrak{g}_0 orbit of $\xi \mathbf{v}_w$ of the form

$$(8.10) \quad \begin{aligned} \beta_o &= \alpha_j + \dots + \alpha_{\mathbf{k}-1} + 2(\alpha_{\mathbf{k}} + \dots + \alpha_{n-1}) + \alpha_n, \\ \gamma_o &= \alpha_{j+1} + \dots + \alpha_{\mathbf{k}} + 2(\alpha_{\mathbf{k}+1} + \dots + \alpha_{n-1}) + \alpha_n, \end{aligned}$$

for some $j, \mathbf{k} \in J$, satisfying $(\gamma_o, \beta_o) \in \Pi(w)$ and $(\beta_o - \gamma_o)(Z_w) = 2$.

(a.a.2) Manipulations similar the those of (a.a.1) yield an element $\xi_o \mathbf{v}_w$ in the \mathfrak{g}_0 -orbit of $\xi \mathbf{v}_w$ of the form

$$(8.11) \quad \beta_o = 2(\alpha_j + \dots + \alpha_{n-1}) + \alpha_n, \quad \gamma_o = 2(\alpha_{j+1} + \dots + \alpha_{n-1}) + \alpha_n,$$

for some $j \in J$. Note that $(\gamma_o, \beta_o) \in \Pi(w)$ and $(\beta_o - \gamma_o)(Z_w) = 2$.

(*,*) Similar arguments in the remaining cases yield $\xi_o \mathbf{v}_w$ in the \mathfrak{g}_0 -orbit of $\xi \mathbf{v}_w$ with $\xi_o = E_{-\beta_o} \otimes E_{\gamma_o}$ and $(\gamma_o, \beta_o) \in \Pi(w)$ of the form (8.10) or (8.11). \square

8.5. Spinor variety D_n/P_n . In this section we show that $B_w = R_w$ holds for the varieties in Theorem 6.1(d).

Let $(\gamma, \beta) \in \Pi(w)$. As a lowest $\mathfrak{g}_{0,0}$ -weight β is of one of the following forms

$$(8.12a) \quad \beta = \alpha_{j_r} + \cdots + \alpha_n \quad (n-1 \in J), \quad \beta = \alpha_{j_r} + \cdots + \alpha_{n-2} + \alpha_n,$$

$$(8.12b) \quad \beta = \alpha_{j_r} + \cdots + \alpha_{j_s-1} + 2(\alpha_{j_s} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n,$$

$$(8.12c) \quad \beta = \alpha_{j_s-1} + 2(\alpha_{j_s} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \quad (j_s - 1 \notin J),$$

for some $j_r, j_s \in J = \{j_1, \dots, j_p\}$. (Cf. the proof of Lemma 3.15 and Corollary 3.17.) Similarly, as a $\mathfrak{g}_{0,0}$ -highest weight, γ is of one of the following forms

$$(8.13a) \quad \gamma = \alpha_{j_t+1} + \cdots + \alpha_n \quad \text{or} \quad \alpha_{j_t+1} + \cdots + \alpha_{n-2} + \alpha_n \quad (n-1 \in J),$$

$$(8.13b) \quad \gamma = \alpha_{j_t+1} + \cdots + \alpha_{j_u} + 2(\alpha_{j_u+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n,$$

$$(8.13c) \quad \gamma = \alpha_{j_t+1} + 2(\alpha_{j_t+2} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n \quad (j_t + 1 \notin J),$$

for some $j_t, j_u \in J \cup \{0\}$, with $j_0 := 0$. The requirement that $\beta - \gamma$ be a sum of positive roots, but not a root itself, rules out the second root in (8.12a). The conditions $j_s - 1 \notin J$ in (8.12c) and $j_t + 1 \notin J$ in (8.13c) hold for all X_w satisfying H_+ ; see Theorem 6.1(d).

Lemma 8.2 holds for $G = D_n$; see the end of this section. The condition $(\beta - \gamma)(Z_w) = 2$ implies that the $\beta - \gamma$ is of one of the two following forms

$$\alpha_j + \alpha_{j'} \quad \text{or} \quad \alpha_{j-1} + 2\alpha_j + \alpha_{j+1},$$

with $j \neq j' \in J$. Thus,

$$(8.14a) \quad \mathbf{b} = \{-\alpha_j, -\alpha_{j'}\} \quad \text{or}$$

$$(8.14b) \quad \mathbf{b} = \{-\alpha_{j-1} - \alpha_j, -\alpha_j - \alpha_{j+1}\}.$$

By Remark 7.13 the weight space $\mathbf{I}_w^{\gamma-\beta}$ is one-dimensional and spanned by $\mathbf{b} \cdot \mathbf{v}_w$. As in Sections 8.2 & 8.4, the statements (8.3) and (8.5) hold.

Suppose $j = j_t < j' = j_u$. First suppose that \mathbf{b} is of the form (8.14a). Then γ is of the form (8.13b). The pair μ, ν of (8.5) exists if and only if either

$$\begin{aligned} 1 < j_{t+1} - j_t \quad \text{or} \quad 1 < j_{u+1} - j_u & \quad (\text{when } \gamma(Z_w) = \mathbf{a}); \\ 1 < j_t - j_{t-1} \quad \text{or} \quad 1 < j_u - j_{u-1} & \quad (\text{when } \gamma(Z_w) = \mathbf{a} - 1). \end{aligned}$$

(If $t = 1$, then $j_0 := 0$. If $u = |J|$, then $j_{u+1} := n - 2$.) Next suppose \mathbf{b} is of the form (8.14b). Then γ is of the form (8.13c). The pair μ, ν of (8.5) exists if and only if

$$\begin{aligned} 2 < j_{t+1} - j_t & \quad (\text{when } \gamma(Z_w) = \mathbf{a}); \\ 2 < j_t - j_{t-1} & \quad (\text{when } \gamma(Z_w) = \mathbf{a} - 1). \end{aligned}$$

When comparing these inequalities to Theorem 6.1(d), it is helpful make the following observations:

- (i) Suppose $n - 1 \notin J$. In this case $\gamma(Z_w) = 2(\mathbf{p} - t)$. If $\gamma(Z_w) = \mathbf{a} - 1$, then $t = \mathbf{p} - r + 1$. If $\gamma(Z_w) = \mathbf{a}$, then $t = \mathbf{p} - r$.
- (ii) Assume $n - 1 \in J$. In this case $\gamma(Z_w) = 2(\mathbf{p} - t) - 1$. If $\gamma(Z_w) = \mathbf{a} - 1$, then $t = \mathbf{p} - r$. If $\gamma(Z_w) = \mathbf{a}$, then $t = \mathbf{p} - r - 1$.

Modulo Lemma 8.2, this establishes the proposition.

Proof of Lemma 8.2 for $G = D_n$. The proof is similar to those for $G = A_n$ (Section 8.2) and $G = C_n$ (Section 8.4). Details are left to the reader. \square

8.6. The exceptional CHSS. It remains to show that $B_w = R_w$ holds for the varieties in Tables 4 & 5.

Lemma 8.2 does not hold for the exceptional CHSS. Rather than argue as in Sections 8.2, 8.4 & 8.5, we verified (with the assistance of LiE [16]) that for each $(\gamma, \beta) \in \Pi(w)$, the corresponding $\xi_{\mathbf{v}_w}$ contains a nontrivial component in some $\mathbf{I}_{w'} \neq \mathbf{I}_w$ with $|w'| = |w|$. Schur rigidity then follows from Theorem 6.1(e), Proposition 7.5 and Lemmas 7.8 and 7.9. (Indeed we can select w' so that $\mathbf{v}_{w'} = E_{-\beta_o} \wedge (E_{\gamma_o} \lrcorner \mathbf{v}_w)$, where γ_o and β_o are respectively highest and lowest $\mathfrak{g}_{0,0}$ -weights, $\gamma_o(Z_w) = \mathbf{a}$ and $\beta_o(Z_w) = \mathbf{a} + 1$. From here, it is then not difficult to see – in each case – that $\xi_{\mathbf{v}_w}$ contains a nontrivial component in $\mathbf{I}_{w'}$.)

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